

Clifford Valued Differential Forms, Algebraic Spinor Fields, Gravitation, Electromagnetism and ”Unified” Theories*

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Abstract

In this paper, we show how to describe the general theory of a linear metric compatible connection with the theory of Clifford valued differential forms. This is done by realizing that for each spacetime point the Lie algebra of Clifford bivectors is isomorphic to the Lie algebra of $Sl(2, \mathbb{C})$. In that way, the pullback of the linear connection under a trivialization of the bundle is represented by a Clifford valued 1-form. That observation makes it possible to realize that Einstein’s gravitational theory can be formulated in a way which is similar to a $Sl(2, \mathbb{C})$ gauge theory. Some aspects of such approach are discussed. Also, the theory of the covariant spinor derivative of spinor fields is introduced in a novel way, allowing for a physical interpretation of some rules postulated for that covariant spinor derivative in the standard theory of these objects. We use our methods to investigate some *polemical* issues in gravitational theories and in particular we scrutinize a supposedly ”unified” field theory of gravitation and electromagnetism proposed by M. Sachs and recently used in a series of

papers by other authors. Our results show that Sachs did not attain his objective and that recent papers based on that theory are ill conceived and completely invalid both as Mathematics and Physics.

1 Introduction

In this paper we introduce the concept of Clifford valued differential forms¹, mathematical entities which are sections of $\mathcal{Cl}(TM) \otimes \bigwedge T^*M$. We show how with the aid of this concept we can produce a very beautiful *description* of the theory of linear connections, where the representative of a given linear connection in a given gauge is represented by a *bivector* valued 1-form. The notion of an exterior covariant differential and exterior covariant derivative of sections of $\mathcal{Cl}(TM) \otimes \bigwedge T^*M$ is crucial for our program and is thus discussed in details. Our *natural* definitions (to be compared with other approaches on related subjects, as described, e.g., in [18, 19, 50, 52, 72, 74, 98]) parallel in a noticeable way the formalism of the theory of connections in principal bundles and their associated covariant derivative operators acting on associated vector bundles. We identify Cartan *curvature 2-forms* and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in complete analogy with equations of gauge theories. This immediately suggests to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past (see e.g., [22, 24]). Our methodology suggest new ways of taking advantage of such a formulation, but this is postpone for a later paper. Here, our investigation of the $Sl(2, \mathbb{C})$ nonhomogeneous gauge equation for the curvature bivector is restricted to the relationship between that equation and Sachs theory [90, 91, 92] and the problem of the energy-momentum 'conservation' in General Relativity.

We recall also the concept of covariant derivatives of (*algebraic*) spinor fields in our formalism, where these objects are represented as sections of real spinor bundles² and study how this theory has as matrix representative the standard two components spinor fields (*dotted* and *undotted*) already introduced long ago, see, e.g., [22, 75, 76, 77]. What is *new* here is that we identify that in the theory of algebraic spinor fields the *realization* of some rules which are used in the standard formulation of the matrix spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null, imply some constraints, with admit a very interesting geometrical interpretation. Indeed, a possible realization of that rules in the Clifford bundle formalism is one where the vector fields defining a global tetrad $\{\mathbf{e}_a\}$ must be such that $D_{\mathbf{e}_0}\mathbf{e}_0 = 0$, i.e., \mathbf{e}_0 is a geodesic reference frame and along each one of its integral lines, say σ , and the \mathbf{e}_d ($d = 1, 2, 3$) are Fermi transported, i.e., they are not rotating relative to the local gyroscope

¹Analogous, but non equivalent concepts have been introduced in [30, 103, 102]. In particular [30] is a very complete paper using cliffforms, i.e., forms with values in a abstract Clifford algebra.

²Real Spinor fields have been introduced by Hestenes in [55], but a rigorous theory of that objects in a Lorentzian spacetime has only recently been achieved [67, 83].

axes. For the best of our knowledge this important fact is here disclosed for the first time.

We use the Clifford bundle formalism and the theory of Clifford valued differential forms to analyze some polemic issues in presentations of gravitational theory and some other theories. In particular, we scrutinized Sachs "unified" theory as described recently in [92] and originally introduced in [90]. We show that unfortunately there are some *serious* mathematical errors in Sachs theory. To start, he identified erroneously his basic variables q_μ as being quaternion fields over a Lorentzian spacetime. Well, they are *not*. The real mathematical structure of these objects is that they are matrix representations of particular sections of the even Clifford bundle of multivectors $\mathcal{Cl}(TM)$ (called paravector fields in mathematical literature) as we proved in section 2. Next we show that the identification of a 'new' antisymmetric field in his theory is indeed nothing more than the identification of some combinations of the curvature bivectors³, an object that appears naturally when we try to formulate Einstein's gravitational theory as a $Sl(2, \mathbb{C})$ gauge theory. In that way, any tentative of identifying such an object with any kind of *electromagnetic field* as did by Sachs in [90, 91, 92] is clearly *wrong*. We note that recently in a series of papers, Evans&AIAS group ([1]-[15],[33]-[37],[27][38]-[42]) uses Sachs theory in order to justify some very *odd* facts, which must be denounced. Indeed, we recall that:

(i) On March 26 2002, the United States Patent and Trademark Office (USPTO) in Washington issued US Patent no. 6,362,718 for a Motionless Electromagnetic Generator (MEG). This would be 'remarkable' device has been projected by retired lieutenant colonel Tom E. Bearden of Alabama and collaborators. They claimed MEG produces more output energy than the input energy used for its functioning!

Of course, nobody could think that the officers at the US Patent office do *not* know the law of energy-momentum conservation, which in general prevents all Patent offices to veto all free energy machines, and indeed that energy momentum conservation law has been used since a long time ago as a *golden rule*.

So, affording a patent to that device must have a reason. A possible one is that the patent officers must somehow been convinced that there are theoretical reasons for the functioning of MEG. How, did the patent officers get convinced?

We think that the answer can be identified in a long list of papers published in respectable (?) Physics journals signed by Evan&AIAS group and quoted above⁴. There, they claimed that using Sachs theory there is a 'nat-

³The curvature bivectors are physically and mathematically equivalent to the Cartan curvature 2-forms, since they carry the same information. This statement will become obvious from our study in section 4.

⁴Note that Bearden is one of the members of the AIAS group. We mention also that in the AIAS website the following people among others are listed as emeritus fellows of the Foundation: Prof. Alwyn van der Merwe, Univ. of Denver, Colorado, USA, Prof. Mendel Sachs, SUNY, Buffalo, USA, Prof. Jean Pierre Vigi r, Institut Henri Poincar , France. Well, van der Merwe is editor of *Foundations of Physics* and *Foundations of Physics Letters*, Sachs is one of the authors we criticize here and Vigi r is on the editorial board of *Physics Letters A* for decades and is one of the AIAS authors. This eventually could explain how AIAS got their papers published...

ural' justification for an entity that they called the \mathbf{B}_3 field and that appears (according to them) in their 'new' $O(3)$ electrodynamics and 'unified' field theory. According to them, the \mathbf{B}_3 field is to be identified with \mathbf{F}_{12} , where $\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu}$ (see Eq.(70) below) is a mathematical object that Sachs identified in [90, 91, 92] with an electromagnetic field after 'taking the trace in the spinor indices'. Evans&AIAS group claim to explain the operation of MEG. It simply pumps energy from the \mathbf{F}_{12} existing in spacetime. However, the Mathematics and Physics of Evans&AIAS used in their papers are unfortunately only a pot pourri of nonsense as we already demonstrated elsewhere⁵ and more below. This, of course invalidate any theoretical justification for the patent.

It would be great if the officers of USPTO would know enough Mathematics and Physics in order to reject immediately the *theoretical* explanations offered by the MEG inventors. But that unfortunately was not the case, because it seems that the knowledge of Mathematics and Physics of that officers was no great than the knowledge of these disciplines by the referees of the Evans&AIAS papers.

Of course, theoretical explanations apart and the authors prejudices it can happen that MEG *works*. However, having followed with interest in the internet⁶ the work of supposedly MEG builders, we arrived at the conclusion that MEG did not work until now, and all claims of its inventors and associates are simply due to *wrong* experimental measurements. And, of course, that must also been the case with the USPTO officers, if they did realize any single experiment on the MEG device. And indeed, this may be really the case, for in a recent article [64] we are informed that in August last year the Commissioner of Patents, Nicholas P. Godici informed that it was a planned a re-examination of the MEG patent. We do not know what happened since then.

(ii) Now, is energy-momentum conservation a *trustworthy* law of the physical world? To answer that question we discuss in this paper the shameful problem of the energy-momentum 'conservation' in General Relativity.

Yes, in General Relativity there are *no* conservation laws of energy, momentum and angular momentum *in general*, and this fact must be clear once and for ever for all (even for school boys, that are in general fooled in reading science books for laymen).

To show this result in an economic and transparent way a presentation of Einstein's gravitational theory is given in terms of tetrads fields, which has a very elegant description in terms of the calculus in the Clifford bundle $\mathcal{Cl}(T^*M)$ described in Appendix. Using that toll, we recall also the *correct* wave like equations solved by the tetrad fields⁷ θ^a in General Relativity. This has been done here in order to complete the debunking of recent Evans&AIAS papers ([27],[38]-[42]) claiming to have achieved (yet) another 'unified' field theory. Indeed, we show that, as it is the case with almost all other papers written by those authors, these new ones are again a compendium of very bad Mathematics

⁵For more details on the absurdities propagated by Evans&AIAS in ISI indexed journals and books see [25, 85]. The second citation is a reply to Evans' paper [38].

⁶See http://groups.yahoo.com/group/free_energy/.

⁷The set $\{\theta^a\}$ is the dual basis of $\{e_a\}$.

and Physics.

2 Spacetime, Pauli and Quaternion Algebras

In this section we recall very well known facts concerning three special *real* Clifford algebras, namely, the *spacetime* algebra $\mathbb{R}_{1,3}$, the *Pauli* algebra $\mathbb{R}_{3,0}$ and the *quaternion* algebra $\mathbb{R}_{0,2} = \mathbb{H}$ and the relation between them.⁸

2.1 Spacetime Algebra

We define the spacetime algebra $\mathbb{R}_{1,3}$ as being the Clifford algebra associated with Minkowski vector space $\mathbb{R}^{1,3}$, which is a four dimensional real vector space, equipped with a Lorentzian bilinear form

$$\boldsymbol{\eta} : \mathbb{R}^{1,3} \times \mathbb{R}^{1,3} \rightarrow \mathbb{R}. \quad (1)$$

Let $\{\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$ be an arbitrary orthonormal basis of $\mathbb{R}^{1,3}$, i.e.,

$$\boldsymbol{\eta}(\mathbf{m}_\mu, \mathbf{m}_\nu) = \eta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu = 1, 2, 3 \\ 0 & \text{if } \mu \neq \nu \end{cases} \quad (2)$$

As usual we resume Eq.(2) writing $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. We denote by $\{\mathbf{m}^0, \mathbf{m}^1, \mathbf{m}^2, \mathbf{m}^3\}$ the *reciprocal* basis of $\{\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3\}$, i.e., $\boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}_\nu) = \delta_\nu^\mu$. We have in obvious notation $\boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}^\nu) = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

The spacetime algebra $\mathbb{R}_{1,3}$ is generate by the following algebraic fundamental relation

$$\mathbf{m}^\mu \mathbf{m}^\nu + \mathbf{m}^\nu \mathbf{m}^\mu = 2\eta^{\mu\nu}. \quad (3)$$

We observe that (as with the conventions fixed in the Appendix) in the above formula and in all the text the Clifford product is denoted by *juxtaposition* of symbols. $\mathbb{R}_{1,3}$ as a vector space over the real field is isomorphic to the

exterior algebra $\bigwedge \mathbb{R}^{1,3} = \sum_{j=0}^4 \bigwedge^j \mathbb{R}^{1,3}$ of $\mathbb{R}^{1,3}$. We code that information writing

$\bigwedge \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}_{1,3}$. Also, $\bigwedge^0 \mathbb{R}^{1,3} \equiv \mathbb{R}$ and $\bigwedge^1 \mathbb{R}^{1,3} \equiv \mathbb{R}^{1,3}$. We identify the exterior product of vectors by

$$\mathbf{m}^\mu \wedge \mathbf{m}^\nu = \frac{1}{2} (\mathbf{m}^\mu \mathbf{m}^\nu - \mathbf{m}^\nu \mathbf{m}^\mu), \quad (4)$$

and also, we identify the scalar product by

$$\boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}^\nu) = \frac{1}{2} (\mathbf{m}^\mu \mathbf{m}^\nu + \mathbf{m}^\nu \mathbf{m}^\mu). \quad (5)$$

⁸This material is treated in details e.g, in the books [56, 60, 78, 79]. See also [44, 45, 46, 47, 68, 69, 70].

Then we can write

$$\mathbf{m}^\mu \mathbf{m}^\nu = \boldsymbol{\eta}(\mathbf{m}^\mu, \mathbf{m}^\nu) + \mathbf{m}^\mu \wedge \mathbf{m}^\nu. \quad (6)$$

From the observations given in the Appendix it follows that an arbitrary element $\mathbf{C} \in \mathbb{R}_{1,3}$ can be written as sum of *nonhomogeneous multivectors*, i.e.,

$$\mathbf{C} = s + c_\mu \mathbf{m}^\mu + \frac{1}{2} c_{\mu\nu} \mathbf{m}^\mu \mathbf{m}^\nu + \frac{1}{3!} c_{\mu\nu\rho} \mathbf{m}^\mu \mathbf{m}^\nu \mathbf{m}^\rho + p \mathbf{m}^5 \quad (7)$$

where $s, c_\mu, c_{\mu\nu}, c_{\mu\nu\rho}, p \in \mathbb{R}$ and $c_{\mu\nu}, c_{\mu\nu\rho}$ are completely antisymmetric in all indices. Also $\mathbf{m}^5 = \mathbf{m}^0 \mathbf{m}^1 \mathbf{m}^2 \mathbf{m}^3$ is the generator of the pseudo scalars. As matrix algebra we have that $\mathbb{R}_{1,3} \simeq \mathbb{H}(2)$, the algebra of the 2×2 quaternionic matrices.

2.2 Pauli Algebra

Now, the Pauli algebra $\mathbb{R}_{3,0}$ is the Clifford algebra associated with the Euclidean vector space $\mathbb{R}^{3,0}$, equipped as usual, with a positive definite bilinear form. As a matrix algebra we have that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$, the algebra of 2×2 complex matrices. Moreover, we recall that $\mathbb{R}_{3,0}$ is isomorphic to the even subalgebra of the spacetime algebra, i.e., writing $\mathbb{R}_{1,3} = \mathbb{R}_{1,3}^{(0)} \oplus \mathbb{R}_{1,3}^{(1)}$ we have,

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)}. \quad (8)$$

The isomorphism is easily exhibited by putting $\boldsymbol{\sigma}^i = \mathbf{m}^i \mathbf{m}^0$, $i = 1, 2, 3$. Indeed, with $\delta^{ij} = \text{diag}(1, 1, 1)$, we have

$$\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j + \boldsymbol{\sigma}^j \boldsymbol{\sigma}^i = 2\delta^{ij}, \quad (9)$$

which is the fundamental relation defining the algebra $\mathbb{R}_{3,0}$. Elements of the Pauli algebra will be called Pauli numbers⁹. As vector space we have that $\bigwedge \mathbb{R}^{3,0} \hookrightarrow \mathbb{R}_{3,0} \subset \mathbb{R}_{1,3}$. So, any Pauli number can be written as

$$\mathbf{P} = s + p^i \boldsymbol{\sigma}^i + \frac{1}{2} p_{ij} \boldsymbol{\sigma}^i \boldsymbol{\sigma}^j + p \mathbf{I}, \quad (10)$$

where $s, p_i, p_{ij}, p \in \mathbb{R}$ and $p_{ij} = -p_{ji}$ and also

$$\mathbf{I} = \boldsymbol{\sigma}^1 \boldsymbol{\sigma}^2 \boldsymbol{\sigma}^3 = \mathbf{m}^5. \quad (11)$$

Note that $\mathbf{I}^2 = -1$ and that \mathbf{I} commutes with any Pauli number. We can trivially verify that

$$\begin{aligned} \boldsymbol{\sigma}^i \boldsymbol{\sigma}^j &= \mathbf{I} \varepsilon_k^{ij} \boldsymbol{\sigma}^k + \delta^{ij}, \\ [\boldsymbol{\sigma}^i, \boldsymbol{\sigma}^j] &\equiv \boldsymbol{\sigma}^i \boldsymbol{\sigma}^j - \boldsymbol{\sigma}^j \boldsymbol{\sigma}^i = 2\boldsymbol{\sigma}^i \wedge \boldsymbol{\sigma}^j = 2\mathbf{I} \varepsilon_k^{ij} \boldsymbol{\sigma}^k. \end{aligned} \quad (12)$$

⁹Sometimes they are also called ‘complex quaternions’. This last terminology will be obvious in a while.

In that way, writing $\mathbb{R}_{3,0} = \mathbb{R}_{3,0}^{(0)} + \mathbb{R}_{3,0}^{(1)}$, any Pauli number can be written as

$$\mathbf{P} = \mathbf{Q}_1 + \mathbf{I}\mathbf{Q}_2, \quad \mathbf{Q}_1 \in \mathbb{R}_{3,0}^{(0)}, \quad \mathbf{I}\mathbf{Q}_2 \in \mathbb{R}_{3,0}^{(1)} \quad (13)$$

with

$$\begin{aligned} \mathbf{Q}_1 &= a_0 + a_k(\mathbf{I}\boldsymbol{\sigma}^k), \quad a_0 = s, \quad a_k = \frac{1}{2}\varepsilon_k^{ij}p_{ij}, \\ \mathbf{Q}_2 &= \mathbf{I}(b_0 + b_k(\mathbf{I}\boldsymbol{\sigma}^k)), \quad b_0 = p, \quad b_k = -p_k. \end{aligned} \quad (14)$$

2.3 Quaternion Algebra

Eqs.(14) show that the quaternion algebra $\mathbb{R}_{0,2} = \mathbb{H}$ can be identified as the even subalgebra of $\mathbb{R}_{3,0}$, i.e.,

$$\mathbb{R}_{0,2} = \mathbb{H} \simeq \mathbb{R}_{3,0}^{(0)}. \quad (15)$$

The statement is obvious once we identify the basis $\{1, \hat{i}, \hat{j}, \hat{k}\}$ of \mathbb{H} with

$$\{1, \mathbf{I}\boldsymbol{\sigma}^1, \mathbf{I}\boldsymbol{\sigma}^2, \mathbf{I}\boldsymbol{\sigma}^3\}, \quad (16)$$

which are the generators of $\mathbb{R}_{3,0}^{(0)}$. We observe moreover that the even subalgebra of the quaternions can be identified (in an obvious way) with the complex field, i.e., $\mathbb{R}_{0,2}^{(0)} \simeq \mathbb{C}$.

Returning to Eq.(10) we see that any $\mathbf{P} \in \mathbb{R}_{3,0}$ can also be written as

$$\mathbf{P} = \mathbf{P}_1 + \mathbf{I}\mathbf{L}_2, \quad (17)$$

where

$$\begin{aligned} \mathbf{P}_1 &= (s + p^k \boldsymbol{\sigma}_k) \in \bigwedge^0 \mathbb{R}^{3,0} \oplus \bigwedge^1 \mathbb{R}^{3,0} \equiv \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0}, \\ \mathbf{I}\mathbf{L}_2 &= \mathbf{I}(p + l^k \boldsymbol{\sigma}_k) \in \bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0}, \end{aligned} \quad (18)$$

with $l_k = -\varepsilon_k^{ij}p_{ij} \in \mathbb{R}$. The important fact that we want to recall here is that the subspaces $(\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ and $(\bigwedge^2 \mathbb{R}^{3,0} \oplus \bigwedge^3 \mathbb{R}^{3,0})$ do not close separately any algebra. In general, if $\mathbf{A}, \mathbf{C} \in (\mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0})$ then

$$\mathbf{AC} \in \mathbb{R} \oplus \bigwedge^1 \mathbb{R}^{3,0} \oplus \bigwedge^2 \mathbb{R}^{3,0}. \quad (19)$$

To continue, we introduce

$$\boldsymbol{\sigma}_i = \mathbf{m}_i \mathbf{m}_0 = -\boldsymbol{\sigma}^i, \quad i = 1, 2, 3. \quad (20)$$

Then, $\mathbf{I} = -\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3$ and the basis $\{1, \hat{i}, \hat{j}, \hat{k}\}$ of \mathbb{H} can be identified with $\{1, -\mathbf{I}\boldsymbol{\sigma}_1, -\mathbf{I}\boldsymbol{\sigma}_2, -\mathbf{I}\boldsymbol{\sigma}_3\}$.

Now, we already said that $\mathbb{R}_{3,0} \simeq \mathbb{C}(2)$. This permit us to represent the Pauli numbers by 2×2 complex matrices, in the usual way ($i = \sqrt{-1}$). We write $\mathbb{R}_{3,0} \ni \mathbf{P} \mapsto P \in \mathbb{C}(2)$, with

$$\begin{aligned} \sigma^1 &\mapsto \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma^2 &\mapsto \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma^3 &\mapsto \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (21)$$

2.4 Minimal left and right ideals in the Pauli Algebra and Spinors

It is not our intention to present the details of algebraic spinor theory here (see, e.g., [49, 80, 60]). However, we will need to recall some facts. The elements $\mathbf{e}_\pm = \frac{1}{2}(1 + \sigma_3) = \frac{1}{2}(1 + \mathbf{m}_3 \mathbf{m}_0) \in \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}$, $\mathbf{e}_\pm^2 = \mathbf{e}_\pm$ are minimal idempotents. They generate the minimal left and right ideals

$$\mathbf{I}_\pm = \mathbb{R}_{1,3}^{(0)} \mathbf{e}_\pm, \quad \mathbf{R}_\pm = \mathbf{e}_\pm \mathbb{R}_{1,3}^{(0)}. \quad (22)$$

From now on we write $\mathbf{e} = \mathbf{e}_+$. It can be easily shown (see below) that, e.g., $\mathbf{I} = \mathbf{I}_+$ has the structure of a 2-dimensional vector space over the complex field [49, 80], i.e., $\mathbf{I} \simeq \mathbb{C}^2$. The elements of the vector space \mathbf{I} are called algebraic *contravariant undotted spinors* and the elements of \mathbb{C}^2 are the usual *contravariant undotted spinors* used in physics textbooks. They carry the $D^{(\frac{1}{2}, 0)}$ representation of $Sl(2, \mathbb{C})$ [63]. If $\varphi \in \mathbf{I}$ we denote by $\varphi \in \mathbb{C}^2$ the usual matrix representative¹⁰ of φ is

$$\varphi = \begin{pmatrix} \varphi^1 \\ \varphi^2 \end{pmatrix}, \quad \varphi^1, \varphi^2 \in \mathbb{C}. \quad (23)$$

We denote by $\dot{\mathbf{I}} = \mathbf{e} \mathbb{R}_{1,3}^{(0)}$ the space of the algebraic covariant dotted spinors. We have the isomorphism, $\dot{\mathbf{I}} \simeq (\mathbb{C}^2)^\dagger \simeq \mathbb{C}_2$, where \dagger denotes Hermitian conjugation. The elements of $(\mathbb{C}^2)^\dagger$ are the usual contravariant spinor fields used in physics textbooks. They carry the $D^{(0, \frac{1}{2})}$ representation of $Sl(2, \mathbb{C})$ [63]. If $\dot{\xi} \in \dot{\mathbf{I}}$ its matrix representation in $(\mathbb{C}^2)^\dagger$ is a row matrix usually denoted by

$$\dot{\xi} = (\xi_1 \quad \xi_2), \quad \xi_1, \xi_2 \in \mathbb{C}. \quad (24)$$

The following representation of $\dot{\xi} \in \dot{\mathbf{I}}$ in $(\mathbb{C}^2)^\dagger$ is extremely convenient. We say that to a covariant undotted spinor ξ there corresponds a covariant dotted spinor $\dot{\xi}$ given by

$$\dot{\mathbf{I}} \ni \dot{\xi} \mapsto \dot{\xi} = \bar{\xi} \varepsilon \in (\mathbb{C}^2)^\dagger, \quad \bar{\xi}_1, \bar{\xi}_2 \in \mathbb{C}, \quad (25)$$

¹⁰The matrix representation of elements of ideals are of course, 2×2 complex matrices (see, [49], for details). It happens that both columns of that matrices have the *same* information and the representation by column matrices is enough here for our purposes.

with

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (26)$$

We can easily find a basis for \mathbf{I} and $\dot{\mathbf{I}}$. Indeed, since $\mathbf{I} = \mathbb{R}_{1,3}^{(0)} \mathbf{e}$ we have that any $\varphi \in \mathbf{I}$ can be written as

$$\varphi = \varphi^1 \vartheta_1 + \varphi^2 \vartheta_2$$

where

$$\begin{aligned} \vartheta_1 &= \mathbf{e}, & \vartheta_2 &= \sigma_1 \mathbf{e} \\ \varphi^1 &= a + \mathbf{i}b, & \varphi^2 &= c + \mathbf{i}d, \quad a, b, c, d \in \mathbb{R}. \end{aligned} \quad (27)$$

Analogously we find that any $\dot{\xi} \in \dot{\mathbf{I}}$ can be written as

$$\begin{aligned} \dot{\xi} &= \xi^1 \mathbf{s}^1 + \xi^2 \mathbf{s}^2 \\ \mathbf{s}^1 &= \mathbf{e}, & \mathbf{s}^2 &= \mathbf{e} \sigma_1. \end{aligned} \quad (28)$$

Defining the mapping

$$\begin{aligned} \iota : \mathbf{I} \otimes \dot{\mathbf{I}} &\rightarrow \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}, \\ \iota(\varphi \otimes \dot{\xi}) &= \varphi \dot{\xi} \end{aligned} \quad (29)$$

we have

$$\begin{aligned} 1 &\equiv \sigma_0 = \iota(\mathbf{s}_1 \otimes \mathbf{s}^1 + \mathbf{s}_2 \otimes \mathbf{s}^2), \\ \sigma_1 &= -\iota(\mathbf{s}_1 \otimes \mathbf{s}^2 + \mathbf{s}_2 \otimes \mathbf{s}^1), \\ \sigma_2 &= \iota[\mathbf{i}(\mathbf{s}_1 \otimes \mathbf{s}^2 - \mathbf{s}_2 \otimes \mathbf{s}^1)], \\ \sigma_3 &= -\iota(\mathbf{s}_1 \otimes \mathbf{s}^1 - \mathbf{s}_2 \otimes \mathbf{s}^2). \end{aligned} \quad (30)$$

From this it follows that we have the identification

$$\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbb{C}(2) = \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}, \quad (31)$$

from where it follows that each Pauli number can be written as an appropriate Clifford product of sums of algebraic contravariant undotted spinors and algebraic covariant dotted spinors, and of course a representative of a Pauli number in \mathbb{C}^2 can be written as an appropriate Kronecker product of a complex column vector by a complex row vector.

Take an arbitrary $\mathbf{P} \in \mathbb{R}_{3,0}$ such that

$$\mathbf{P} = \frac{1}{j!} p^{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j} \sigma_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j}, \quad (32)$$

where $p^{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j} \in \mathbb{R}$ and

$$\sigma_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j} = \sigma_{\mathbf{k}_1} \dots \sigma_{\mathbf{k}_j}, \quad \text{and } \sigma_0 \equiv 1 \in \mathbb{R}. \quad (33)$$

With the identification $\mathbb{R}_{3,0} \simeq \mathbb{R}_{1,3}^{(0)} \simeq \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}}$, we can write also

$$\mathbf{P} = \mathbf{P}^A_{\dot{B}} \iota(\mathbf{s}_A \otimes \mathbf{s}^{\dot{B}}) = \mathbf{P}^A_{\dot{B}} s_B \mathbf{s}^{\dot{B}}, \quad (34)$$

where the $\mathbf{P}^A_{\dot{B}} = \mathbf{X}^A_{\dot{B}} + \mathbf{iY}^A_{\dot{B}}$, $\mathbf{X}^A_{\dot{B}}, \mathbf{Y}^A_{\dot{B}} \in \mathbb{R}$.

Finally, the matrix representative of the Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ is $P \in \mathbb{C}(2)$ given by

$$P = P^A_{\dot{B}} s_A \mathbf{s}^{\dot{B}}, \quad (35)$$

with $P^A_{\dot{B}} \in \mathbb{C}$ and

$$\begin{aligned} s_1 &= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} & s_2 &= \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \\ s^{\dot{1}} &= \begin{pmatrix} 1 & 0 \end{pmatrix} & s^{\dot{2}} &= \begin{pmatrix} 0 & 1 \end{pmatrix}. \end{aligned} \quad (36)$$

It is convenient for our purposes to introduce also covariant undotted spinors and contravariant dotted spinors. Let $\varphi \in \mathbb{C}^2$ be given as in Eq.(23). We define the *covariant* version of undotted spinor $\varphi \in \mathbb{C}^2$ as $\varphi^* \in (\mathbb{C}^2)^t \simeq \mathbb{C}_2$ such that

$$\begin{aligned} \varphi^* &= (\varphi_1, \varphi_2) \equiv \varphi_A s^A, \\ \varphi_A &= \varphi^B \varepsilon_{BA}, \quad \varphi^B = \varepsilon^{BA} \varphi_A, \\ s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \end{aligned} \quad (37)$$

where¹¹ $\varepsilon_{AB} = \varepsilon^{AB} = \text{adiag}(1, -1)$. We can write due to the above identifications that there exists $\varepsilon \in \mathbb{C}(2)$ given by Eq.(26) which can be written also as

$$\varepsilon = \varepsilon^{AB} s_A \boxtimes s_B = \varepsilon_{AB} s^A \boxtimes s^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{i}\sigma_2 \quad (38)$$

where \boxtimes denote the *Kronecker* product of matrices. We have, e.g.,

$$\begin{aligned} s_1 \boxtimes s_2 &= \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ s^1 \boxtimes s^1 &= \begin{pmatrix} 1 & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (39)$$

We now introduce the *contravariant* version of the dotted spinor

$$\dot{\xi} = \begin{pmatrix} \xi_{\dot{1}} & \xi_{\dot{2}} \end{pmatrix} \in \mathbb{C}_2$$

¹¹The symbol *adiag* means the antidiagonal matrix.

as being $\dot{\xi}^* \in \mathbb{C}^2$ such that

$$\begin{aligned}\dot{\xi}^* &= \begin{pmatrix} \xi^{\dot{1}} \\ \xi^{\dot{2}} \end{pmatrix} = \xi^{\dot{A}} s_{\dot{A}}, \\ \xi^{\dot{B}} &= \varepsilon^{\dot{B}\dot{A}} \xi_{\dot{A}}, \quad \xi_{\dot{A}} = \varepsilon_{\dot{B}\dot{A}} \xi^{\dot{B}}, \\ s_{\dot{1}} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_{\dot{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\end{aligned}\tag{40}$$

where $\varepsilon_{\dot{A}\dot{B}} = \varepsilon^{\dot{A}\dot{B}} = \text{adiag}(1, -1)$. We can write due to the above identifications that there exists $\dot{\varepsilon} \in \mathbb{C}(2)$ such that

$$\dot{\varepsilon} = \varepsilon^{\dot{A}\dot{B}} s_{\dot{A}} \boxtimes s_{\dot{B}} = \varepsilon_{\dot{A}\dot{B}} s^{\dot{A}} \boxtimes s^{\dot{B}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon.\tag{41}$$

Also, recall that even if $\{\mathbf{s}_A\}, \{\mathbf{s}_{\dot{A}}\}$ and $\{s^{\dot{A}}\}, \{s^A\}$ are bases of distinct spaces, we can identify their matrix representations, as it is obvious from the above formulas. So, we have $s_A \equiv s_{\dot{A}}$ and also $s^{\dot{A}} = s^A$. This is the reason for the representation of a dotted covariant spinor as in Eq.(25). Moreover, the above identifications permit us to write the *matrix representation* of a Pauli number $\mathbf{P} \in \mathbb{R}_{3,0}$ as, e.g.,

$$P = P_{AB} s^A \boxtimes s^B\tag{42}$$

besides the representation given by Eq.(35).

3 Clifford and Spinor Bundles

3.1 Preliminaries

To characterize in a rigorous mathematical way the *basic* field variables used in Sachs ‘unified’ field theory [90, 91], we shall need to recall some results of the theory of spinor fields on Lorentzian spacetimes. Here we follow the approach given in [83, 67].¹²

Recall that a Lorentzian manifold is a pair (M, g) , where $g \in \text{sec } T^{2,0}M$ is a Lorentzian metric of signature $(1, 3)$, i.e., for all $x \in M$, $T_x M \simeq T_x^* M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.

Recall that a Lorentzian spacetime is a pentuple $(M, g, D, \tau_g, \uparrow)$ where (M, g, τ_g, \uparrow) is an oriented Lorentzian manifold¹³ which is also time oriented by an appropriated equivalence relation¹⁴ (denoted \uparrow) for the timelike vectors at the tangent space $T_x M$, $\forall x \in M$. D is a linear connection for M such that $Dg = 0$, $\Theta(D) = 0$, $\mathcal{R}(D) \neq 0$, where Θ and \mathcal{R} are respectively the torsion and curvature tensors of D .

¹²Another important reference on the subject of spinor fields is [59], which however only deals with the case of spinor fields on Riemannian manifolds.

¹³Oriented by the volume element $\tau_g \in \text{sec } \bigwedge^4 T^* M$.

¹⁴See [93] for details.

Now, Sachs theory uses spinor fields. These objects are sections of so-called *spinor bundles*, which only exist in *spin manifolds*. The ones of interest in Sachs theory are the matrix representation of the bundle of dotted spinor fields, i.e., $S(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{D^{(\frac{1}{2},0)}} \mathbb{C}^2$ and the matrix representation of the bundle of undotted spinor fields (here denoted by) $\bar{S}(M) = P_{\text{Spin}_{1,3}^e}(M) \times_{D^{(0,\frac{1}{2})}} \mathbb{C}^2$. In the previous formula $D^{(\frac{1}{2},0)}$ and $D^{(0,\frac{1}{2})}$ are the two fundamental non equivalent 2-dimensional representations of $Sl(2, \mathbb{C}) \simeq \text{Spin}_{1,3}^e$, the universal covering group of $\text{SO}_{1,3}^e$, the restrict orthochronous Lorentz group. $P_{\text{Spin}_{1,3}^e}(M)$ is a principal bundle called the spin structure bundle¹⁵. We recall that it is a classical result (Geroch theorem [51]) that a 4-dimensional Lorentzian manifold is a spin manifold if and only if $P_{\text{SO}_{1,3}^e}(M)$ has a global section¹⁶, i.e., if there exists a set¹⁷ $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of orthonormal fields defined for all $x \in M$. In other words, in order for spinor fields to exist in a 4-dimensional spacetime the orthonormal frame bundle must be *trivial*.

Now, the so-called tangent (TM) and cotangent (T^*M) bundles, the tensor bundle ($\oplus_{r,s} \otimes_s^r TM$) and the bundle of differential forms for the spacetime are the bundles denoted by

$$\begin{aligned} TM &= P_{\text{SO}_{1,3}^e}(M) \times_{\rho_{1,3}} \mathbb{R}^{1,3}, \quad T^*M = P_{\text{SO}_{1,3}^e}(M) \times_{\rho_{1,3}^*} \mathbb{R}^{1,3}, \\ \oplus_{r,s} \otimes_s^r TM &= P_{\text{SO}_{1,3}^e}(M) \times_{\otimes_s^r \rho_{1,3}} \mathbb{R}^{1,3}, \quad \bigwedge T^*M = P_{\text{SO}_{1,3}^e}(M) \times_{\Lambda_{\rho_{1,3}^*}^k} \bigwedge \mathbb{R}^{1,3}. \end{aligned} \quad (43)$$

In Eqs.(43)

$$\rho_{1,3} : \text{SO}_{1,3}^e \rightarrow \text{SO}^e(\mathbb{R}^{1,3}) \quad (44)$$

is the standard vector representation of $\text{SO}_{1,3}^e$ usually denoted by¹⁸ $D^{(\frac{1}{2}, \frac{1}{2})} = D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})}$ and $\rho_{1,3}^*$ is the dual (vector) representation $\rho_{1,3}^*(l) = \rho_{1,3}(l^{-1})^t$. Also $\otimes_s^r \rho_{1,3}$ and $\Lambda_{\rho_{1,3}^*}^k$ are the induced tensor product and induced exterior power product representations of $\text{SO}_{1,3}^e$. We now briefly recall the definition and some properties of Clifford bundle of multivector fields [83]. We have,

$$\begin{aligned} \mathcal{C}\ell(TM) &= P_{\text{SO}_{1,3}^e}(M) \times_{\mathcal{C}\ell_{\rho_{1,3}}} \mathbb{R}_{1,3} \\ &= P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}. \end{aligned} \quad (45)$$

Now, recall that [60] $\text{Spin}_{1,3}^e \subset \mathbb{R}_{1,3}^{(0)}$. Consider the 2-1 homomorphism $h : \text{Spin}_{1,3}^e \rightarrow \text{SO}_{1,3}^e, h(\pm u) = l$. Then $\mathcal{C}\ell_{\rho_{1,3}}$ is the following representation of

¹⁵It is a covering space of $P_{\text{SO}_{1,3}^e}(M)$. See, e.g., [67] for details. Sections of $P_{\text{Spin}_{1,3}^e}(M)$ are the so-called spin frames, i.e., a pair (Σ, u) where for any $x \in M$, $\Sigma(x)$ is an orthonormal frame and $u(x)$ belongs to the $\text{Spin}_{1,3}^e$. For details see [67, 83, 86].

¹⁶In what follows $P_{\text{SO}_{1,3}^e}(M)$ denotes the principal bundle of oriented *Lorentz tetrads*. We presuppose that the reader is acquainted with the structure of $P_{\text{SO}_{1,3}^e}(M)$, whose sections are the time oriented and oriented orthonormal frames, each one associated by a local trivialization to a *unique* element of $\text{SO}_{1,3}^e(M)$.

¹⁷Called vierbein.

¹⁸See, e.g., [63] if you need details.

$\text{SO}_{1,3}^e$,

$$\begin{aligned} \mathcal{C}\ell_{\rho_{1,3}} &: \text{SO}_{1,3}^e \rightarrow \text{Aut}(\mathbb{R}_{1,3}), \\ \mathcal{C}\ell_{\rho_{1,3}}(L) &= \text{Ad}_u : \mathbb{R}_{1,3} \rightarrow \mathbb{R}_{1,3}, \\ \text{Ad}_u(\mathbf{m}) &= u\mathbf{m}u^{-1} \end{aligned} \quad (46)$$

i.e., it is the standard orthogonal transformation of $\mathbb{R}_{1,3}$ induced by an orthogonal transformation of $\mathbb{R}^{1,3}$. Note that Ad_u act on vectors as the $D^{(\frac{1}{2}, \frac{1}{2})}$ representation of $\text{SO}_{1,3}^e$ and on multivectors as the induced exterior power representation of that group. Indeed, observe, e.g., that for $\mathbf{v} \in \mathbb{R}^{1,3} \subset \mathbb{R}_{1,3}$ we have in standard notation

$$L\mathbf{v} = \mathbf{v}^\nu L_\nu^\mu \mathbf{m}_\mu = \mathbf{v}^\nu u \mathbf{m}_\nu u^{-1} = u \mathbf{v} u^{-1}.$$

The proof of the second line of Eq.(45) is as follows. Consider the representation

$$\begin{aligned} \text{Ad} : \text{Spin}_{1,3}^e &\rightarrow \text{Aut}(\mathbb{R}_{1,3}), \\ \text{Ad}_u : \mathbb{R}_{1,3} &\rightarrow \mathbb{R}_{1,3}, \quad \text{Ad}_u(m) = umu^{-1}. \end{aligned} \quad (47)$$

Since $\text{Ad}_{-1} = 1$ (= identity) the representation Ad descends to a representation of $\text{SO}_{1,3}^e$. This representation is just $\mathcal{C}\ell(\rho_{1,3})$, from where the desired result follows.

Sections of $\mathcal{C}\ell(TM)$ can be called Clifford fields (of multivectors). The sections of the even subbundle $\mathcal{C}\ell^{(0)}(TM) = P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}^{(0)}$ may be called Pauli fields (of multivectors). Define the real spinor bundles

$$\mathcal{S}(M) = P_{\text{Spin}_{1,3}^e}(M) \times_l \mathbf{I}, \quad \dot{\mathcal{S}}(M) = P_{\text{Spin}_{1,3}^e}(M) \times_r \dot{\mathbf{I}} \quad (48)$$

where l stands for a left modular representation of $\text{Spin}_{1,3}^e$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(\frac{1}{2}, 0)}$ representation of $Sl(2, \mathbb{C})$ and r stands for a right modular representation of $\text{Spin}_{1,3}^e$ in $\mathbb{R}_{1,3}$ that mimics the $D^{(0, \frac{1}{2})}$ representation of $Sl(2, \mathbb{C})$.

Also recall that if $\bar{S}(M)$ is the bundle whose sections are the spinor fields $\bar{\varphi} = (\bar{\varphi}_1, \bar{\varphi}_2) = \dot{\varphi}\varepsilon = (\varphi^{\dot{1}}, \varphi^{\dot{2}})$, then it is isomorphic to the space of contravariant dotted spinors. We have,

$$S(M) \simeq P_{\text{Spin}_{1,3}^e}(M) \times_{D^{(\frac{1}{2}, 0)}} \mathbb{C}^2, \quad \dot{S}(M) \simeq P_{\text{Spin}_{1,3}^e}(M) \times_{D^{(0, \frac{1}{2})}} \mathbb{C}_2 \simeq \bar{S}(M), \quad (49)$$

and from our playing with the Pauli algebra and dotted and undotted spinors in section 2 we have that:

$$\mathcal{S}(M) \simeq S(M), \quad \dot{\mathcal{S}}(M) \simeq \dot{S}(M) \simeq \bar{S}(M). \quad (50)$$

Then, we have the obvious isomorphism

$$\begin{aligned} \mathcal{C}\ell^{(0)}(TM) &= P_{\text{Spin}_{1,3}^e}(M) \times_{\text{Ad}} \mathbb{R}_{1,3}^{(0)} \\ &= P_{\text{Spin}_{1,3}^e}(M) \times_{l \otimes r} \mathbf{I} \otimes_{\mathbb{C}} \dot{\mathbf{I}} \\ &= \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M). \end{aligned} \quad (51)$$

Let us now introduce the following bundle,

$$\mathbb{C}\ell^{(0)}(M) = P_{\text{Spin}_{1,3}^c}(M) \times_{D^{(\frac{1}{2},0)} \otimes D^{(0,\frac{1}{2})}} \mathbb{C}(2). \quad (52)$$

It is clear that

$$\mathbb{C}\ell^{(0)}(M) = S(M) \otimes_{\mathbb{C}} \bar{S}(M) \simeq \mathcal{C}\ell^{(0)}(M). \quad (53)$$

Finally, we consider the bundle

$$\mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M \simeq \mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M. \quad (54)$$

Sections of $\mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$ may be called *Pauli valued differential forms* and sections of $\mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$ may be called *matrix Pauli valued differential forms*.

Denote by $\mathcal{C}\ell_{(0,2)}^{(0)}(TM)$ the seven dimensional subbundle $\left(\mathbb{R} \oplus \bigwedge^2 TM\right) \subset \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(TM) \subset \mathcal{C}\ell(TM)$. Now, let $\langle x^\mu \rangle$ be the coordinate functions of a chart of the maximal atlas of M . The fundamental field variable of Sachs theory can be described as

$$\mathbf{Q} = \mathbf{q}_\mu \otimes dx^\mu \equiv q_\mu dx^\mu \in \sec \mathcal{C}\ell_{(0,2)}^{(0)}(TM) \otimes \bigwedge T^*M \subset \sec \mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$$

i.e., a Pauli valued 1-form obeying certain conditions to be presented below. If we work (as Sachs did) with $\mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$, a representative of \mathbf{Q} is $Q \in \sec \mathbb{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$ such that¹⁹

$$Q = q_\mu(x) dx^\mu = h_\mu^{\mathbf{a}}(x) dx^\mu \sigma_{\mathbf{a}}, \quad (55)$$

where $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and σ_j ($j=1, 2, 3$) are the Pauli matrices. We observe that the notation anticipates the fact that in Sachs theory the variables $h_\mu^{\mathbf{a}}(x)$ define the set $\{\theta^{\mathbf{a}}\} \equiv \{\theta^0, \theta^1, \theta^2, \theta^3\}$ with

$$\theta^{\mathbf{a}} = h_\mu^{\mathbf{a}} dx^\mu \in \sec \bigwedge T^*M, \quad (56)$$

which is the dual basis of $\{\mathbf{e}_{\mathbf{a}}\} \equiv \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, $\mathbf{e}_{\mathbf{a}} \in \sec TM$. We denote by $\{e_\mu\} = \{e_0, e_1, e_2, e_3\}$, a coordinate basis associated with the local chart $\langle x^\mu \rangle$ covering $U \subset M$. We have $e_\mu = h_\mu^{\mathbf{a}} \mathbf{e}_{\mathbf{a}} \in \sec TM$, and the set $\{e_\mu\}$ is the dual basis of $\{dx^\mu\} \equiv \{dx^0, dx^1, dx^2, dx^3\}$. We will also use the *reciprocal basis* to a given basis $\{\mathbf{e}_{\mathbf{a}}\}$, i.e., the set $\{\mathbf{e}^{\mathbf{a}}\} \equiv \{\mathbf{e}^0, \mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3\}$, $\mathbf{e}^{\mathbf{a}} \in \sec TM$, with $g(\mathbf{e}_{\mathbf{a}}, \mathbf{e}^{\mathbf{b}}) = \delta_{\mathbf{a}}^{\mathbf{b}}$ and the *reciprocal basis* to $\{\theta^{\mathbf{a}}\}$, i.e., the set $\{\theta_{\mathbf{a}}\} = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, with $\theta_{\mathbf{a}}(\mathbf{e}^{\mathbf{b}}) = \delta_{\mathbf{a}}^{\mathbf{b}}$. Recall that since $\eta_{\mathbf{ab}} = g(\mathbf{e}_{\mathbf{a}}, \mathbf{e}_{\mathbf{b}})$, we have

$$g_{\mu\nu} = g(e_\mu, e_\nu) = h_\mu^{\mathbf{a}} h_\nu^{\mathbf{b}} \eta_{\mathbf{ab}}. \quad (57)$$

¹⁹Note that a bold index (sub or superscript), say \mathbf{a} take the values 0, 1, 2, 3.

To continue, we define the

$$\check{\sigma}_0 = -\sigma_0 \text{ and } \check{\sigma}_j = \sigma_j, j = 1, 2, 3 \quad (58)$$

and

$$\check{Q} = \check{q}_\mu(x) dx^\mu = h_\mu^{\mathbf{a}}(x) dx^\mu \check{\sigma}_{\mathbf{a}}. \quad (59)$$

Also, note that

$$\sigma_{\mathbf{a}} \check{\sigma}_{\mathbf{b}} + \sigma_{\mathbf{b}} \check{\sigma}_{\mathbf{a}} = -2\eta_{\mathbf{ab}}. \quad (60)$$

Readers of Sachs books [90, 92] will recall that he said that Q is a representative of a *quaternion*.²⁰ From our previous discussion we see that this statement is *wrong*.²¹ Sachs identification is a dangerous one, because the quaternions are a division algebra, also-called a noncommutative field or skew-field and objects like $\mathbf{Q} = \mathbf{q}_\mu \otimes dx^\mu \in \sec \mathcal{C}\ell_{(0,2)}^{(0)}(TM) \otimes \bigwedge T^*M \subset \sec \mathcal{C}\ell^{(0)}(TM) \otimes \bigwedge T^*M$ are called *paravector* fields. As it is clear from our discussion they did not close a *division* algebra.

Next we introduce a tensor product of sections $\mathbf{A}, \mathbf{B} \in \sec \mathcal{C}\ell^{(0)}(M) \otimes \bigwedge T^*M$. Before we do that we recall that from now on

$$\{1, \sigma_{\mathbf{k}}, \sigma_{\mathbf{k}_1 \mathbf{k}_2}, \sigma_{\mathbf{123}}\}, \quad (61)$$

refers to a basis of $\mathcal{C}\ell^{(0)}(M)$, i.e., they are fields.²²

Recalling Eq.(33) we introduce the (obvious) notation

$$\mathbf{A} = \frac{1}{j!} a_\mu^{k_1 k_2 \dots k_j} \sigma_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j} dx^\mu, \quad \mathbf{B} = \frac{1}{l!} b_\mu^{k_1 k_2 \dots k_l} \sigma_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_l} dx^\mu, \quad (62)$$

where the $a_\mu^{k_1 k_2 \dots k_j}, b_\mu^{k_1 k_2 \dots k_j}$ are, in general, *real* scalar functions. Then, we define

$$\mathbf{A} \otimes \mathbf{B} = \frac{1}{j!l!} a_\mu^{k_1 k_2 \dots k_j} b_\nu^{p_1 p_2 \dots p_l} \sigma_{\mathbf{k}_1 \mathbf{k}_2 \dots \mathbf{k}_j} \sigma_{\mathbf{p}_1 \mathbf{p}_2 \dots \mathbf{p}_l} dx^\mu \otimes dx^\nu. \quad (63)$$

Let us now compute the tensor product of $\mathbf{Q} \otimes \check{\mathbf{Q}}$ where $\mathbf{Q} \in \sec \mathcal{C}\ell_{(0,2)}^{(0)}(M) \otimes \bigwedge T^*M$. We have,

²⁰Note that Sachs represented Q by dS , which is a very dangerous notation, which we avoid.

²¹Nevertheless the calculations done by Sachs in [90] are correct because he worked always with the matrix representation of \mathbf{Q} . However, his claim of having produce an unified field theory of gravitation and electromagnetism is wrong as we shall prove in what follows.

²²We hope that in using (for symbol economy) the same notation as in section 2 where the $\{1, \sigma_{\mathbf{k}}, \sigma_{\mathbf{k}_1 \mathbf{k}_2}, \sigma_{\mathbf{123}}\}$ is a basis of $\mathbb{R}_{1,3}^{(0)} \simeq \mathbb{R}_{3,0}$ will produce no confusion.

$$\begin{aligned}
\mathbf{Q} \otimes \check{\mathbf{Q}} &= \mathbf{q}_\mu(x) dx^\mu \otimes \check{\mathbf{q}}_\nu(x) dx^\nu = \mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) dx^\mu \otimes dx^\nu \\
&= \mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) \frac{1}{2} (dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu) \\
&\quad + \frac{1}{2} \mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) (dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) \\
&= \frac{1}{2} (\mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) + \mathbf{q}_\nu(x) \check{\mathbf{q}}_\mu(x)) dx^\mu \otimes dx^\nu \\
&\quad + \frac{1}{2} \mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) dx^\mu \wedge dx^\nu \\
&= (-g_{\mu\nu} \sigma_0) dx^\mu \otimes dx^\nu \\
&\quad + \frac{1}{4} (\mathbf{q}_\mu(x) \check{\mathbf{q}}_\nu(x) - \mathbf{q}_\nu(x) \check{\mathbf{q}}_\mu(x)) dx^\mu \wedge dx^\nu \\
&= -g_{\mu\nu} dx^\mu \otimes dx^\nu + \frac{1}{2} \mathbf{F}'_{\mu\nu} dx^\mu \wedge dx^\nu.
\end{aligned} \tag{64}$$

In writing Eq.(64) we have used $dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$. Also, using

$$\begin{aligned}
g_{\mu\nu} &= \eta_{\mathbf{ab}} h_\mu^{\mathbf{a}}(x) h_\nu^{\mathbf{b}}(x), \quad g = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{\mathbf{ab}} \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} \\
\mathbf{F}'_{\mu\nu} &= \mathbf{F}'_{\mu\nu}{}^k \mathbf{i}\sigma_k = -\frac{1}{2} (\varepsilon_{ij}^k h_\mu^i(x) h_\nu^j(x)) \mathbf{i}\sigma_k; \quad i, j, k = 1, 2, 3, \\
\mathbf{F}' &= \frac{1}{2} \mathbf{F}'_{\mu\nu} dx^\mu \wedge dx^\nu = \frac{1}{2} (\mathbf{F}'_{\mu\nu}{}^k \sigma_i \sigma_j) dx^\mu \wedge dx^\nu = (\frac{1}{2} \mathbf{F}'_{\mu\nu}{}^k \mathbf{i}\sigma_k) dx^\mu \wedge dx^\nu \\
&= -\varepsilon_{ij}^k h_\mu^i(x) h_\nu^j(x) dx^\mu \wedge dx^\nu \mathbf{i}\sigma_k \in \sec \bigwedge^2 T^*M \otimes \mathcal{C}\ell_{(2)}^{(0)}(M)
\end{aligned} \tag{65}$$

we can write Eq.(64) as

$$\begin{aligned}
\mathbf{Q} \otimes \check{\mathbf{Q}} &= \mathbf{Q} \overset{s}{\otimes} \check{\mathbf{Q}} + \mathbf{Q} \wedge \check{\mathbf{Q}} \\
&= -g + \mathbf{F}.
\end{aligned} \tag{66}$$

We can also write

$$\mathbf{Q} \otimes \check{\mathbf{Q}} = -\eta_{\mathbf{ab}} \sigma_0 \theta^{\mathbf{a}} \otimes \theta^{\mathbf{b}} + \varepsilon_{ij}^k \mathbf{i}\sigma_k \theta^i \wedge \theta^j. \tag{67}$$

The above formulas show very clearly the mathematical nature of \mathbf{F} , it is a 2-form with values on the subspace of multivector Clifford fields, i.e., $\mathbf{F} : \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell_{(2)}^{(0)}(TM) \subset \mathcal{C}\ell^{(0)}(TM)$. Now, we write the formula for $Q \otimes \check{Q}$ where $Q \in \mathbb{C}(2) \otimes \bigwedge^1 T^*M$ given by Eq.(55) is the matrix representation of $\mathbf{Q} \in \sec \mathcal{C}\ell_{(0,2)}^{(0)}(M) \otimes \bigwedge^1 T^*M$.

We have,

$$\begin{aligned}
Q \otimes \tilde{Q} &= Q \overset{s}{\otimes} \tilde{Q} + Q \wedge \tilde{Q} \\
&= (-g_{\mu\nu} dx^\mu \otimes dx^\nu) \sigma_0 + (\varepsilon_{ij}^k v_\mu^i(x) v_\nu^j(x) dx^\mu \wedge dx^\nu) (-i\sigma_k) \\
&= -g\sigma_0 + \mathbf{F}'^k i\sigma_k,
\end{aligned} \tag{68}$$

with

$$\mathbf{F}'^k = \frac{1}{2} \mathbf{F}'_{\mu\nu}^k dx^\mu \wedge dx^\nu = \varepsilon_{ij}^k v_\mu^i(x) v_\nu^j(x) dx^\mu \wedge dx^\nu. \tag{69}$$

For future reference we also introduce

$$\mathbf{F}'_{\mu\nu} = \mathbf{F}'_{\mu\nu}^k i\sigma_k. \tag{70}$$

3.2 Covariant Derivatives of Spinor Fields

We now briefly recall the concept of covariant spinor derivatives [26, 59, 67, 83]. The idea is the following:

(i) Every connection on the principal bundle of orthonormal frames $P_{\text{SO}_{1,3}^\epsilon}(M)$ determines in a canonical way a unique connection on the principal bundle $P_{\text{Spin}_{1,3}^\epsilon}(M)$.

(ii) Let D be a covariant derivative operator acting on sections of an associate vector bundle to $P_{\text{SO}_{1,3}^\epsilon}(M)$, say, the tensor bundle τM and let D^s be the corresponding covariant spinor derivative acting on sections of associate vector bundles to $P_{\text{Spin}_{1,3}^\epsilon}(M)$, say, e.g., the spinor bundles where $\mathcal{P}(M)$ may be called *Pauli spinor bundle*. Of course, $\mathcal{P}(M) \simeq \mathcal{C}\ell^{(0)}(M)$. The matrix representations of the above bundles are:

$$\begin{aligned}
S(M) &= P_{\text{Spin}_{1,3}^\epsilon}(M) \times_{D^{(\frac{1}{2}0)}} \mathbb{C}^2, \quad \dot{S}(M) = P_{\text{Spin}_{1,3}^\epsilon}(M) \times_{D^{(0,\frac{1}{2})}} \mathbb{C}_2 \\
P(M) &= S(M) \otimes \dot{S}(M) = P_{\text{Spin}_{1,3}^\epsilon}(M) \times_{D^{(\frac{1}{2}0)} \otimes D^{(0,\frac{1}{2})}} \mathbb{C}^2 \otimes \mathbb{C}_2,
\end{aligned} \tag{71}$$

and $P(M)$ may be called *matrix Pauli spinor bundle*. Of course, $P(M) \simeq \mathbb{C}\ell^{(0)}(M)$.

(iv) We have for $\mathbf{T} \in \sec \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(M)$ and $\dot{\boldsymbol{\xi}} \in \sec \dot{S}(M)$, $\mathbf{P} \in \sec \mathcal{P}(M)$ and $\mathbf{v} \in \sec TM$. Then,

$$\begin{aligned}
D_{\mathbf{v}}^s(\mathbf{T} \otimes \dot{\boldsymbol{\xi}}) &= D_{\mathbf{v}} \mathbf{T} \otimes \dot{\boldsymbol{\xi}} + \mathbf{T} \otimes D_{\mathbf{v}}^s \dot{\boldsymbol{\xi}}, \\
D_{\mathbf{v}}^s(\mathbf{T} \otimes \dot{\boldsymbol{\xi}}) &= D_{\mathbf{v}} \mathbf{T} \otimes \dot{\boldsymbol{\xi}} + \mathbf{T} \otimes D_{\mathbf{v}}^s \dot{\boldsymbol{\xi}},
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
D_{\mathbf{v}}\mathbf{T} &= \partial_{\mathbf{v}}\mathbf{T} + \frac{1}{2}[\omega_{\mathbf{v}}, \mathbf{T}], \\
D_{\mathbf{v}}^s\xi &= \partial_{\mathbf{v}}\xi + \frac{1}{2}\omega_{\mathbf{v}}\xi, \\
D_{\mathbf{v}}^s\dot{\xi} &= \partial_{\mathbf{v}}\dot{\xi} - \frac{1}{2}\dot{\xi}\omega_{\mathbf{v}}, \\
D_{\mathbf{v}}P &= \partial_{\mathbf{v}}P + \frac{1}{2}\omega_{\mathbf{v}}P - \frac{1}{2}P\omega_{\mathbf{v}} = \partial_{\mathbf{v}}P + \frac{1}{2}[\omega_{\mathbf{v}}, P].
\end{aligned} \tag{73}$$

(v) For $\mathbf{T} \in \sec \bigwedge TM \hookrightarrow \mathcal{C}\ell^{(0)}(TM)$ and $\xi \in \sec S(M)$, $\bar{\xi} \in \sec \bar{S}(M)$, $P \in \sec P(M)$ and $\mathbf{v} \in \sec TM$, we have

$$\begin{aligned}
D_{\mathbf{v}}^s(\mathbf{T} \otimes \xi) &= D_{\mathbf{v}}\mathbf{T} \otimes \xi + \mathbf{T}D_{\mathbf{v}}^s\xi, \\
D_{\mathbf{v}}^s(\mathbf{T} \otimes \bar{\xi}) &= D_{\mathbf{v}}\mathbf{T} \otimes \bar{\xi} + \mathbf{T}D_{\mathbf{v}}^s\bar{\xi}
\end{aligned} \tag{74}$$

and

$$\begin{aligned}
D_{\mathbf{v}}\mathbf{T} &= \partial_{\mathbf{v}}\mathbf{T} + \frac{1}{2}[\omega_{\mathbf{v}}, \mathbf{T}], \\
D_{\mathbf{v}}^s\xi &= \partial_{\mathbf{v}}\xi + \frac{1}{2}\Omega_{\mathbf{v}}\xi, \\
D_{\mathbf{v}}^s\dot{\xi} &= \partial_{\mathbf{v}}\dot{\xi} - \frac{1}{2}\dot{\xi}\Omega_{\mathbf{v}}, \\
D_{\mathbf{v}}P &= \partial_{\mathbf{v}}P + \frac{1}{2}\Omega_{\mathbf{v}}P - \frac{1}{2}P\Omega_{\mathbf{v}} = \partial_{\mathbf{v}}P + \frac{1}{2}[\Omega_{\mathbf{v}}, P].
\end{aligned} \tag{75}$$

In the above equations $\omega_{\mathbf{v}} \in \sec \mathcal{C}\ell^{(0)}(TM)$ and $\Omega_{\mathbf{v}} \in \sec P(M)$. Writing as usual, $\mathbf{v} = v^{\mathbf{a}}\mathbf{e}_{\mathbf{a}}$, $D_{\mathbf{e}_{\mathbf{a}}}e^{\mathbf{b}} = -\omega_{\mathbf{a}\mathbf{c}}^{\mathbf{b}}e^{\mathbf{c}}$, $\omega_{\mathbf{a}\mathbf{b}\mathbf{c}} = -\omega_{\mathbf{c}\mathbf{b}\mathbf{a}} = \eta_{\mathbf{a}\mathbf{d}}\omega_{\mathbf{b}\mathbf{c}}^{\mathbf{d}}$, $\omega_{\mathbf{b}}^{\mathbf{a}\mathbf{c}} = -\omega_{\mathbf{b}}^{\mathbf{c}\mathbf{a}}$, $\sigma_{\mathbf{b}} = \mathbf{e}_{\mathbf{b}}\mathbf{e}_0$ and²³ $i = -\sigma_1\sigma_2\sigma_3$, we have

$$\begin{aligned}
\omega_{\mathbf{e}_{\mathbf{a}}} &= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{b}\mathbf{c}}\mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{c}} = \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{b}\mathbf{c}}\mathbf{e}_{\mathbf{b}} \wedge \mathbf{e}_{\mathbf{c}} \\
&= \frac{1}{2}\omega_{\mathbf{a}}^{\mathbf{b}\mathbf{c}}\sigma_{\mathbf{b}}\overset{\vee}{\sigma}_{\mathbf{c}} \\
&= \frac{1}{2}(-2\omega_{\mathbf{a}}^{0i}\sigma_i + \omega_{\mathbf{a}}^{ji}\sigma_i\sigma_j) \\
&= \frac{1}{2}(-2\omega_{\mathbf{a}}^{0i}\sigma_i - i\varepsilon_{ij}^k\omega_{\mathbf{a}}^{ji}\sigma_k) = \Omega_{\mathbf{a}}^{\mathbf{b}}\sigma_{\mathbf{b}}.
\end{aligned} \tag{76}$$

Note that the $\Omega_{\mathbf{a}}^{\mathbf{b}}$ are ‘formally’ complex numbers. Also, observe that we can write for the ‘formal’ Hermitian conjugate $\omega_{\mathbf{e}_{\mathbf{a}}}^{\dagger}$ of $\omega_{\mathbf{e}_{\mathbf{a}}}$ of

$$\omega_{\mathbf{e}_{\mathbf{a}}}^{\dagger} = -\mathbf{e}^0\omega_{\mathbf{e}_{\mathbf{a}}}\mathbf{e}^0. \tag{77}$$

²³Have in mind that i is a *Clifford field* here.

Also, write $\Omega_{\mathbf{e}_a}$ for the matrix representation of $\omega_{\mathbf{e}_a}$, i.e.,

$$\Omega_{\mathbf{e}_a} = \Omega_{\mathbf{a}}^{\mathbf{b}} \sigma_{\mathbf{b}},$$

where $\Omega_{\mathbf{a}}^{\mathbf{b}}$ are complex numbers with the same coefficients as the ‘formally’ complex numbers $\Omega_{\mathbf{a}}^{\mathbf{b}}$. We can easily verify that

$$\Omega_{\mathbf{e}_a} = \varepsilon \Omega_{\mathbf{e}_a}^{\dagger} \varepsilon. \quad (78)$$

We can prove the third line of Eq.(75) as follows. First take the Hermitian conjugation of the second line of Eq.(75), obtaining

$$D_{\mathbf{v}} \bar{\xi} = \partial_{\mathbf{v}} \bar{\xi} + \frac{1}{2} \bar{\xi} \Omega_{\mathbf{v}}^{\dagger}.$$

Next multiply the above equation on the left by ε and recall that $\dot{\xi} = \bar{\xi} \varepsilon$ and Eq.(78). We get

$$\begin{aligned} D_{\mathbf{v}} \dot{\xi} &= \partial_{\mathbf{v}} \dot{\xi} - \frac{1}{2} \dot{\xi} \varepsilon \Omega_{\mathbf{v}}^{\dagger} \varepsilon \\ &= D_{\mathbf{v}} \dot{\xi} = \partial_{\mathbf{v}} \dot{\xi} - \frac{1}{2} \dot{\xi} \Omega_{\mathbf{v}}. \end{aligned}$$

Note that this is compatible with the identification $\mathcal{C}\ell^{(0)}(TM) \simeq \mathcal{S}(M) \otimes_{\mathbb{C}} \dot{\mathcal{S}}(M)$ and $\mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \dot{S}(M)$.

Note moreover that if $\mathbf{q}_{\mu} = e_{\mu} \mathbf{e}_0 = h_{\mu}^{\mathbf{a}} \mathbf{e}_a \mathbf{e}_0 = h_{\mu}^{\mathbf{a}} \sigma_{\mathbf{a}} \in \mathcal{C}\ell^{(0)}(TM) \simeq S(M) \otimes_{\mathbb{C}} \dot{S}(M)$ we have,

$$D_{\mathbf{v}} \mathbf{q}_{\mu} = \partial_{\mathbf{v}} q_{\mu} + \frac{1}{2} \omega_{\mathbf{v}} \mathbf{q}_{\mu} + \frac{1}{2} \mathbf{q}_{\mu} \omega_{\mathbf{v}}^{\dagger}. \quad (79)$$

For $q_{\mu} = h_{\mu}^{\mathbf{a}} \sigma_{\mathbf{a}} \in \sec \mathbb{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \bar{S}(M)$, the matrix representative of the \mathbf{q}_{μ} we have for any vector field $\mathbf{v} \in \sec TM$

$$D_{\mathbf{v}} q_{\mu} = \partial_{\mathbf{v}} q_{\mu} + \frac{1}{2} \Omega_{\mathbf{v}} q_{\mu} + \frac{1}{2} q_{\mu} \Omega_{\mathbf{v}}^{\dagger} \quad (80)$$

which is the equation used by Sachs for the *spinor* covariant derivative of his ‘quaternion’ fields. Note that M. Sachs in [90] introduced also a kind of total covariant derivative for his ‘quaternion’ fields. That ‘derivative’ denoted in this text by $D_{\mathbf{v}}^{\mathbf{S}}$ will be discussed below.

3.3 Geometrical Meaning of $D_{e_{\nu}} q_{\mu} = \Gamma_{\nu\mu}^{\alpha} q_{\alpha}$

We recall that Sachs wrote ²⁴ that

$$D_{e_{\nu}} q_{\mu} = \Gamma_{\nu\mu}^{\alpha} q_{\alpha}, \quad (81)$$

²⁴See Eq.(3.69) in [90].

where $\Gamma_{\nu\mu}^\alpha$ are the connection coefficients of the coordinate basis $\{e_\mu\}$, i.e.,

$$D_{e_\nu} e_\mu = \Gamma_{\nu\mu}^\alpha e_\alpha \quad (82)$$

How, can Eq.(81) be true? Well, let us calculate $D_{e_\nu} \mathbf{q}_\mu$ in $\mathcal{C}\ell(TM)$. We have,

$$\begin{aligned} D_{e_\nu} \mathbf{q}_\mu &= D_{e_\nu} (e_\mu \mathbf{e}_0) \\ &= (D_{e_\nu} e_\mu) \mathbf{e}_0 + e_\mu (D_{e_\nu} \mathbf{e}_0) \\ &= \Gamma_{\nu\mu}^\alpha \mathbf{q}_\alpha + e_\mu (D_{e_\nu} \mathbf{e}_0). \end{aligned} \quad (83)$$

So, Eq.(81) follows if, and only if

$$D_{e_\nu} \mathbf{e}_0 = 0. \quad (84)$$

To understand the physical meaning of Eq.(84) let us recall the following. In relativity theory reference frames are represented by time like vector fields $Z \in \sec TM$ pointing to the future [84, 93]. If we write the $\alpha_{\mathbf{Z}} = g(\mathbf{Z}, \cdot) \in \bigwedge^1 T^*M$ for the physically equivalent 1-form field we have the well known *decomposition*

$$D\alpha_{\mathbf{Z}} = \mathbf{a}_{\mathbf{Z}} \otimes \alpha_{\mathbf{Z}} + \varpi_{\mathbf{Z}} + \sigma_{\mathbf{Z}} + \frac{1}{3} E_{\mathbf{Z}} \mathbf{p}, \quad (85)$$

where

$$\mathbf{p} = g - \alpha_{\mathbf{Z}} \otimes \alpha_{\mathbf{Z}} \quad (86)$$

is called the projection tensor (and gives the metric of the rest space of an instantaneous observer [93]), $\mathbf{a}_{\mathbf{Z}} = g(D_{\mathbf{Z}} \mathbf{Z}, \cdot)$ is the (form) acceleration of \mathbf{Z} , $\varpi_{\mathbf{Z}}$ is the rotation of \mathbf{Z} , $\sigma_{\mathbf{Z}}$ is the shear of \mathbf{Z} and $E_{\mathbf{Z}}$ is the expansion ratio of \mathbf{Z} . In a coordinate chart (U, x^μ) , writing $\mathbf{Z} = Z^\mu \partial / \partial x^\mu$ and $\mathbf{p} = (g_{\mu\nu} - Z_\mu Z_\nu) dx^\mu \otimes dx^\nu$ we have

$$\begin{aligned} \varpi_{\mathbf{Z}\mu\nu} &= Z_{[\alpha;\beta]} p_\mu^\alpha p_\nu^\beta, \\ \sigma_{\mathbf{Z}\alpha\beta} &= [Z_{(\mu;\nu)} - \frac{1}{3} E_{\mathbf{Z}} h_{\mu\nu}] p_\alpha^\mu p_\beta^\nu, \\ E_{\mathbf{Z}} &= Z^\mu{}_{;\mu}. \end{aligned} \quad (87)$$

Now, in Special Relativity where the space time manifold is the structure $(M = \mathcal{R}^4, g = \eta, D^\eta, \tau_\eta, \uparrow)^{25}$ an *inertial reference frame (IRF)* $\mathbf{I} \in \sec TM$ is defined by $D^\eta \mathbf{I} = 0$. We can show very easily (see, e.g., [93]) that in General Relativity Theory (*GRT*) where each gravitational field is modelled by a spacetime²⁶ $(M, g, D, \tau_g, \uparrow)$ there is *in general* no shear free frame ($\sigma_\Omega = 0$) on any open neighborhood U of any given spacetime point. The reason is clear

²⁵ η is a constant metric, i.e., there exists a chart $\langle x^\mu \rangle$ of $M = \mathcal{R}^4$ such that $\eta(\partial/\partial x^\mu, \partial/\partial x^\nu) = \eta_{\mu\nu}$, the numbers $\eta_{\mu\nu}$ forming a diagonal matrix with entries $(1, -1, -1, -1)$. Also, D^η is the Levi-Civita connection of η .

²⁶More precisely, by a diffeomorphism equivalence class of Lorentzian spacetimes.

in local coordinates $\langle x^\mu \rangle$ covering U . Indeed, $\sigma_\Omega = 0$ implies five independent conditions on the components of the frame Ω . Then, we arrive at the conclusion that in a general spacetime model²⁷ there is no frame $\Omega \in \sec TU \subset \sec TM$ satisfying $D\Omega = 0$, and in general there is no *IRF* in any model of *GRT*.

The following question arises naturally: which characteristics a reference frame on a *GRT* spacetime model must have in order to reflect as much as possible the properties of an *IRF* of *SRT*?

The answer to that question [84] is that there are two kind of frames in *GRT* such that each frame in one of these classes share some important aspects of the *IRFs* of *SRT*. Both concepts are important and it is important to distinguish between them in order to avoid misunderstandings. These frames are the *pseudo inertial reference frame (PIRF)* and the local Lorentz reference frames (*LLRFs*), but we do not need to enter the details here.

On the open set $U \subset M$ covered by a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of M multiplying Eq.(84) by $h^\nu_{\mathbf{a}}$ such that $\mathbf{e}_{\mathbf{a}} = h^\nu_{\mathbf{a}} e_\nu$, we get

$$D_{\mathbf{e}_{\mathbf{a}}} \mathbf{e}_0 = 0; \quad \mathbf{a} = 0, 1, 2, 3. \quad (88)$$

Then, it follows that

$$D_X \mathbf{e}_0 = 0, \quad \forall X \in \sec TM \quad (89)$$

which characterizes \mathbf{e}_0 as an inertial frame. This imposes several restrictions on the spacetime described by the theory. Indeed, if \mathbf{Ric} is the Ricci tensor of the manifold modeling spacetime, we have²⁸

$$\mathbf{Ric}(\mathbf{e}_0, X) = 0, \quad \forall X \in \sec TM. \quad (90)$$

In particular, this condition cannot be realized in Einstein-de Sitter spacetime. This fact is completely hidden in the matrix formalism used in Sachs theory, where no restriction on the spacetime manifold (besides the one of being a spin manifold) need to be imposed.

3.4 Geometrical Meaning of $D_{e_\mu} \sigma_i = 0$ in General Relativity

We now discuss what happens in the usual theory of dotted and undotted two component *matrix* spinor fields in general relativity, as described, e.g., in [22, 75, 76]. In that formulation it is postulated that the covariant spinor derivative of Pauli matrices must satisfy

$$D_{e_\mu} \sigma_i = 0, \quad i=1, 2, 3 \quad (91)$$

²⁷We take the opportunity to correct an statement in [84]. There it is stated that in General Relativity there are no inertial frames. Of, course, the correct statement is that in a general spacetime model there are in general no inertial frames. But, of course, there are spacetime models where there exist frames $\Omega \in \sec TU \subset \sec TM$ satisfying $D\Omega = 0$. See below.

²⁸See, exercise 3.2.12 of [93].

Eq.(91) translate in our formalism as

$$D_{e_\mu} \sigma_i = D_{e_\mu} (\mathbf{e}_i \mathbf{e}_0) = 0. \quad (92)$$

Differently from the case of Sachs theory, Eq.(92) can be satisfied if

$$D_{e_\mu} \mathbf{e}_i = \mathbf{e}_i (D_{e_\mu} \mathbf{e}_0) \mathbf{e}_0 \quad (93)$$

or, writing $D_{e_\mu} \mathbf{e}_a = \omega_{\mu a}^b \mathbf{e}_b$,

$$\omega_{\mu i}^b = \mathbf{e}^b \lrcorner (\omega_{\mu 0}^a \mathbf{e}_i \mathbf{e}_a \mathbf{e}_0). \quad (94)$$

This certainly implies some restrictions on possible spacetime models, but that is the price in order to have spinor fields. At least we do not need to necessarily have $D\mathbf{e}_0 = 0$.

We analyze some possibilities of satisfying Eq.(91)

(i) Suppose that \mathbf{e}_0 satisfy $D_{e_\mu} \mathbf{e}_0 = 0$, i.e., $D\mathbf{e}_0 = 0$. Then, a necessary and sufficient condition for the validity of Eq.(92) is that

$$D_{e_\mu} \mathbf{e}_i = 0. \quad (95)$$

Multiplying Eq.(95) by h_a^μ we get

$$D_{\mathbf{e}_a} \mathbf{e}_i = 0, \quad i=1, 2, 3; \quad \mathbf{a} = 0, 1, 2, 3 \quad (96)$$

In particular,

$$D_{\mathbf{e}_0} \mathbf{e}_i = 0, \quad i=1, 2, 3 \quad (97)$$

Eq.(97) means that the fields \mathbf{e}_i following each integral line of \mathbf{e}_0 are Fermi transported²⁹ [93]. Physicists interpret that equation saying that the \mathbf{e}_i are physically realizable by gyroscopic axes, which gives the local standard of no rotation.

The above conclusion sounds fine. However it follows from Eq.(89) and Eq.(96) that

$$D_{\mathbf{e}_a} \mathbf{e}_b = 0, \quad \mathbf{a}=0,1,2,3; \quad \mathbf{b} = 0, 1, 2, 3. \quad (98)$$

Recalling that existence of spinor fields implies that $\{\mathbf{e}_a\}$ is a global tetrad [51], Eq.(98) implies that the connection D must be teleparallel. Then, under the above conditions the curvature tensor of a spacetime admitting spinor fields must be *null*. This, is in particular, the case of special relativity.

(ii) Suppose now that \mathbf{e}_0 is a geodesic frame, i.e., $D_{\mathbf{e}_0} \mathbf{e}_0 = 0$. Then, $h_0^\nu D_{e_\nu} \mathbf{e}_0 = 0$ and Eq. (93) implies only that

$$D_{\mathbf{e}_0} \mathbf{e}_i = 0; \quad i=1, 2, 3 \quad (99)$$

and we do not have *any* inconsistency. If we take an integral line of \mathbf{e}_0 , say γ , then the set $\{\mathbf{e}_a|_\gamma\}$ may be called an *inertial moving frame* along γ . The set

²⁹An original approach to the Fermi transport using Clifford bundle methods has been given in [82]. There an equivalent spinor equation to the famous Darboux equations of differential geometry is derived.

$\{\mathbf{e}_a|_\gamma\}$ is also *Fermi* transported and since γ is a geodesic worldline they define the standard of *no* rotation along γ .

In conclusion, a consistent definition of spinor fields in general relativity using the Clifford and spin bundle formalism of this paper needs triviality of the frame bundle, i.e., existence of a global tetrad, say $\{\mathbf{e}_a\}$ and validity of Eq.(93). A nice physical interpretation follows moreover if the tetrad satisfies

$$D_{\mathbf{e}_0}\mathbf{e}_a = 0; \quad a=0,1,2,3. \quad (100)$$

Of course, as it is the case in Sachs theory, the matrix formulation of spinor fields do not impose any constraints in the possible spacetime models, besides the one needed for the existence of a spinor structure. Saying that we have an important comment.

3.5 Covariant Derivative of the Dirac Gamma Matrices

If we use a real spin bundle where we can formulate the Dirac equation, e.g., one where the typical fiber is the ideal of (algebraic) Dirac spinors, i.e., the ideal generated by a idempotent $\frac{1}{2}(1 + E_0)$, $E_0 \in \mathbb{R}_{1,3}$, then no restriction is imposed on the global tetrad field $\{\mathbf{e}_a\}$ defining the spinor structure of spacetime (see [83, 67]). In particular, since

$$D_{\mathbf{e}_a}\mathbf{e}_b = \omega_{ab}^c \mathbf{e}_c, \quad (101)$$

we have,

$$D_{\mathbf{e}_a}\mathbf{e}_b = \frac{1}{2}[\omega_{\mathbf{e}_a}, \mathbf{e}_b] \quad (102)$$

Then,

$$\omega_{ab}^c \mathbf{e}_c - \frac{1}{2}\omega_{\mathbf{e}_a}\mathbf{e}_b + \frac{1}{2}\mathbf{e}_b\omega_{\mathbf{e}_a} = 0. \quad (103)$$

The matrix representation of the real spinor bundle, of course, sends $\{\mathbf{e}_a\} \mapsto \{\gamma_a\}$, where the γ_a 's are the standard representation of the Dirac matrices. Then, the matrix translation of Eq.(103) is

$$\omega_{ab}^c \gamma_c - \frac{1}{2}\omega_{\mathbf{e}_a}\gamma_b + \frac{1}{2}\gamma_b\omega_{\mathbf{e}_a} = 0. \quad (104)$$

For the matrix elements γ_{bB}^A we have

$$\omega_{ab}^c \gamma_{cB}^A - \frac{1}{2}\omega_{\mathbf{e}_a}^A \gamma_{bB}^C + \frac{1}{2}\gamma_{bB}^A \omega_{\mathbf{e}_a}^C = 0. \quad (105)$$

In [26] this last equation is confused with the covariant derivative of γ_{cB}^A . Indeed in an exercise in problem 4, Chapter Vbis [26] ask one to prove that

$$\boxed{\nabla_{\mathbf{e}_b}\gamma_{cB}^A = \omega_{ab}^c \gamma_{cB}^A - \frac{1}{2}\omega_{\mathbf{e}_a}^A \gamma_{bB}^C + \frac{1}{2}\gamma_{bB}^A \omega_{\mathbf{e}_a}^C = 0.}$$

Of course, the first member of the above equation does not define any covariant derivative operator. Confusions as that one appears over and over again in the literature, and of course, is also present in Sachs theory in a small modified form, as shown in the next subsection.

3.6 $D_{e_\nu}^{\mathbf{S}} q_\mu = 0$

Now, taking into account Eq.(80) and Eq.(81) we can write:

$$\partial_\nu \mathbf{q}_\mu + \frac{1}{2} \boldsymbol{\omega}_\nu \mathbf{q}_\mu + \frac{1}{2} \mathbf{q}_\mu \boldsymbol{\omega}_\nu - \Gamma_{\nu\mu}^\alpha \mathbf{q}_\alpha = 0. \quad (106)$$

Sachs defined

$$D_{e_\nu}^{\mathbf{S}} \mathbf{q}_\mu = \partial_\nu \mathbf{q}_\mu + \frac{1}{2} \boldsymbol{\omega}_\nu \mathbf{q}_\mu + \frac{1}{2} \mathbf{q}_\mu \boldsymbol{\omega}_\nu - \Gamma_{\nu\mu}^\alpha \mathbf{q}_\alpha \quad (107)$$

from where

$$D_{e_\nu}^{\mathbf{S}} \mathbf{q}_\mu = 0. \quad (108)$$

Of course, the matrix representation of the last two equations are:

$$\begin{aligned} D_{e_\nu}^{\mathbf{S}} q_\mu &= \partial_\nu q_\mu + \frac{1}{2} \Omega_\nu q_\mu + \frac{1}{2} q_\mu \Omega_\nu^\dagger - \Gamma_{\nu\mu}^\alpha q_\alpha. \\ D_{e_\nu}^{\mathbf{S}} q_\mu &= 0. \end{aligned} \quad (109)$$

Sachs call ³⁰ $D_{e_\nu}^{\mathbf{S}} q_\mu$ the covariant derivative of a q_μ field. The nomination is an *unfortunate* one, since the equation $D_{e_\nu}^{\mathbf{S}} q_\mu = 0$ is a *trivial* identity and do not introduce any new connection in the game.³¹

After this long exercise we can derive easily all formulas in chapters 3-6 of [90] without using any matrix representation at all. In particular, for future reference we collect some formulas,

$$\begin{aligned} \mathbf{q}^\mu \tilde{\mathbf{q}}_\mu &= -4, \quad q^\mu \tilde{q}_\mu = -4\sigma_0 \\ \mathbf{q}_\rho^\mu \boldsymbol{\omega} \tilde{\mathbf{q}}_\mu &= 0, \quad q^\mu \Omega_\rho \tilde{q}_\mu = 0, \\ \boldsymbol{\omega}_\rho &= -\frac{1}{2} \tilde{\mathbf{q}}_\mu (\partial_\rho \mathbf{q}^\mu + \Gamma_{\rho\tau}^\mu \mathbf{q}^\tau), \quad \Omega_\rho = -\frac{1}{2} \tilde{q}_\mu (\partial_\rho q^\mu + \Gamma_{\rho\tau}^\mu q^\tau) \end{aligned} \quad (110)$$

Before we proceed, it is important to keep in mind that our ‘normalization’ of $\boldsymbol{\omega}_\rho$ (and of Ω_ρ) here *differs* from Sachs one by a factor of 1/2. We prefer our normalization, since it is more natural and avoid factors of 2 when we perform contractions.

Before we discuss the equations of Sachs theory we think it is worth, using Clifford algebra methods, to present a formulation of Einstein’s gravitational theory which resembles a gauge theory with group $Sl(2, \mathbb{C})$ as the gauge group. This formulation will then be compared with Sachs theory. Our formulation permits to prove that contrary to his claims in [90, 91] he did not produce any unified field theory of gravitation and electromagnetism.

³⁰See Eq.(3.69) in [90].

³¹The equation $D_\nu^{\mathbf{S}} \mathbf{q}_\mu = 0$ (or its matrix representation) is a reminiscence of an analogous equation for the components of tetrad fields often printed in physics textbooks and confused with the metric compatibility condition of the connection. See, e.g., comments on page 76 of [52].

4 Recall of Some Facts of the Theory of Linear Connections

4.1 Preliminaries

In the general theory of connections [26, 57] a connection is a 1-form in the cotangent space of a principal bundle, with values in the Lie algebra of a gauge group. In order to develop a theory of a linear connection³²

$$\overset{\blacktriangle}{\omega} \in \sec T^*P_{\text{SO}_{1,3}^e}(M) \otimes \mathfrak{sl}(2, \mathbb{C}), \quad (111)$$

with an *exterior* covariant derivative operator acting on sections of associated vector bundles to the principal bundle $P_{\text{SO}_{1,3}^e}(M)$ which reproduces moreover the well known results obtained with the usual covariant derivative of tensor fields in the base manifold, we need to introduce the concept of a *soldering* form

$$\overset{\blacktriangle}{\theta} \in \sec T^*P_{\text{SO}_{1,3}^e}(M) \otimes \mathbb{R}^{1,3}. \quad (112)$$

Let be $U \subset M$ and π_1, π_2 respectively the projections of $T^*P_{\text{SO}_{1,3}^e}(M) \otimes \mathbb{R}^{1,3}$ and $P_{\text{SO}_{1,3}^e}(M)$ to M , naturally associated to the projection π of $P_{\text{SO}_{1,3}^e}(M)$. Let

$$\begin{aligned} \varsigma_1 : U &\rightarrow \pi_1^{-1}(U) \subset T^*P_{\text{SO}_{1,3}^e}(M) \otimes \mathbb{R}^{1,3}, \\ \varsigma_2 : U &\rightarrow \pi_2^{-1}(U) \subset T^*P_{\text{SO}_{1,3}^e}(M) \otimes \mathfrak{sl}(2, \mathbb{C}), \end{aligned} \quad (113)$$

be two cross sections. We are interested in the study of the pullbacks $\omega = \varsigma_2^* \overset{\blacktriangle}{\omega}$ and $\theta = \varsigma_1^* \overset{\blacktriangle}{\theta}$ once we give a local trivialization of the respective bundles. As it is well known [57], we have in a local chart $\langle x^\mu \rangle$ covering U ,

$$\theta = e_\mu \otimes dx^\mu \equiv e_\mu dx^\mu \in \sec TM \otimes \bigwedge^1 T^*M. \quad (114)$$

Now, we give the Clifford algebra structure to the tangent bundle, thus generating the Clifford bundle $\mathcal{Cl}(TM) = \bigcup_x \mathcal{Cl}_x(M)$, with $\mathcal{Cl}_x(M) \simeq \mathbb{R}_{1,3}$ introduced in Appendix A.

We recall moreover, a well known result [60], namely, that for each $x \in U \subset M$ the bivectors of $\mathcal{Cl}(T_x M)$ generate under the product defined by the commutator, the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. We thus are lead to define the representatives in $\mathcal{Cl}(TM) \otimes \bigwedge T^*M$ for θ and for the the pullback ω of the connection *in a*

³²In words, $\overset{\blacktriangle}{\omega}$ is a 1-form in the cotangent space of the bundle of orthonormal frames with values in the Lie algebra $\mathfrak{so}_{1,3}^e \simeq \mathfrak{sl}(2, \mathbb{C})$ of the group $\text{SO}_{1,3}^e(M)$.

given gauge (that we represent with the same symbols):

$$\begin{aligned}
\boldsymbol{\theta} &= e_\mu dx^\mu = \mathbf{e}_a \theta^a \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M, \\
\boldsymbol{\omega} &= \frac{1}{2} \omega_a^{bc} \mathbf{e}_b \mathbf{e}_c \theta^a \\
&= \frac{1}{2} \omega_a^{bc} (\mathbf{e}_b \wedge \mathbf{e}_c) \otimes \theta^a \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M.
\end{aligned} \tag{115}$$

Before we continue we must recall that whereas $\boldsymbol{\theta}$ is a true tensor, $\boldsymbol{\omega}$ is not a true tensor, since as it is well known, its ‘components’ do not have the tensor transformation properties. Note that the ω_a^{bc} are the ‘components’ of the connection defined by

$$D_{\mathbf{e}_a} e^b = -\omega_{ac}^b e^c, \quad \omega_{ac}^b = -\omega_{ca}^b, \tag{116}$$

where $D_{\mathbf{e}_a}$ is a metric compatible covariant derivative operator³³ defined on the tensor bundle, that naturally acts on $\mathcal{C}\ell(TM)$ (see, e.g., [28]). Objects like $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ will be called Clifford valued differential forms (or Clifford valued forms, for short)³⁴, and in the next sections we give a detailed account of the algebra and calculus of that objects. But, before we start this project we need to recall some concepts of the theory of linear connections.

4.2 Exterior Covariant Differential

One of our objectives is to show how to describe, with our formalism an exterior covariant differential (*EXCD*) which acts naturally on sections of Clifford valued differential forms (i.e., sections of $\sec \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$) and which *mimics* the action of the pullback of the exterior covariant derivative operator acting on sections of a vector bundle associated to the principal bundle $P_{\text{SO}_{1,3}^\epsilon}(M)$, once a linear metrical compatible connection is given. Our motivation for the definition of the *EXCD* is that with it, the calculations of curvature bivectors, Bianchi identities, etc., use always the same formula. Of course, we compare our definition, with other definitions of analogous, but distinct concepts, already used in the literature, showing where they differ from ours, and why we think that ours seems more appropriate. In particular, with the *EXCD* and its associated *extended* covariant derivative (*ECD*) we can write Einstein’s equations in such a way that the resulting equation looks like an equation for a gauge theory of the group $Sl(2, \mathbb{C})$. To achieve our goal, we recall below the well known definition of the exterior covariant differential \mathbf{d}^E acting on arbitrary sections of a

³³After section 2.5, $D_{\mathbf{e}_a}$ refers to the Levi-Civita covariant derivative operator.

³⁴Analogous, but non equivalent concepts have been introduced in [30, 103, 105, 102]. In particular [30] introduce cliffords, i.e., forms with values in a abstract (internal) Clifford algebra $\mathbb{R}_{p,q}$ associated with a pair (\mathbb{R}^n, g) , where $n = p + q$ and g is a bilinear form of signature (p, q) in \mathbb{R}^n . These objects *differ* from the Clifford valued differential forms used in this text., whith dispenses any abstract (internal) space.

vector bundle $E(M)$ (associated to $P_{\text{SO}_{1,3}}(M)$ and having as typical fiber a l -dimensional real vector space) and on $\text{end}E(M) = E(M) \otimes E^*(M)$, the bundle of endomorphisms of $E(M)$. We recall also the concept of absolute differential acting on sections of the tensor bundle, for the particular case of $\bigwedge^l TM$.

Definition 1 *The exterior covariant differential operator \mathbf{d}^E acting on sections of $E(M)$ and $\text{end}E(M)$ is the mapping*

$$\mathbf{d}^E : \text{sec } E(M) \rightarrow \text{sec } E(M) \otimes \bigwedge^1 T^*M, \quad (117)$$

such that for any differentiable function $f : M \rightarrow \mathbb{R}$, $A \in \text{sec } E(M)$ and any $F \in \text{sec}(\text{end}E(M) \otimes \bigwedge^p T^*M)$, $G \in \text{sec}(\text{end}E(M) \otimes \bigwedge^q T^*M)$ we have:

$$\begin{aligned} \mathbf{d}^E(fA) &= df \otimes A + f\mathbf{d}^E A, \\ \mathbf{d}^E(F \otimes_\wedge A) &= \mathbf{d}^E F \otimes_\wedge A + (-1)^p F \otimes_\wedge \mathbf{d}^E A, \\ \mathbf{d}^E(F \otimes_\wedge G) &= \mathbf{d}^E F \otimes_\wedge G + (-1)^p F \otimes_\wedge \mathbf{d}^E G. \end{aligned} \quad (118)$$

In Eq.(118), writing $F = F^a \otimes f_a^{(p)}$, $G = G^b \otimes g_b^{(q)}$ where $F^a, G^b \in \text{sec}(\text{end}E(M))$, $f_a^{(p)} \in \text{sec } \bigwedge^p T^*M$ and $g_b^{(q)} \in \text{sec } \bigwedge^q T^*M$ we have

$$\begin{aligned} F \otimes_\wedge A &= (F^a \otimes f_a^{(p)}) \otimes_\wedge A, \\ F \otimes_\wedge G &= (F^a \otimes f_a^{(p)}) \otimes_\wedge G^b \otimes g_b^{(q)}. \end{aligned} \quad (119)$$

In what follows, in order to simplify the notation we eventually use when there is no possibility of confusion, the simplified (sloppy) notation

$$\begin{aligned} (F^a A) \otimes f_a^{(p)} &\equiv (F^a A) f_a^{(p)}, \\ (F^a \otimes f_a^{(p)}) \otimes_\wedge G^b \otimes g_b^{(q)} &= (F^a G^b) f_a^{(p)} \wedge g_b^{(q)}, \end{aligned} \quad (120)$$

where $F^a A \in \text{sec } E(M)$ and $F^a G^b$ means the composition of the respective endomorphisms.

Let $U \subset M$ be an open subset of M , $\langle x^\mu \rangle$ a coordinate functions of a maximal atlas of M , $\{e_\mu\}$ a coordinate basis of $TU \subset TM$ and $\{s_{\mathbf{K}}\}$, $\mathbf{K}=1, 2, \dots, l$ a basis for any $\text{sec } E(U) \subset \text{sec } E(M)$. Then, a basis for any section of $E(M) \otimes \bigwedge^1 T^*M$ is given by $\{s_{\mathbf{K}} \otimes dx^\mu\}$.

Definition 2 *The covariant derivative operator $D_{e_\mu} : \text{sec } E(M) \rightarrow \text{sec } E(M)$ is given by*

$$\mathbf{d}^E A \doteq (D_{e_\mu} A) \otimes dx^\mu, \quad (121)$$

where, writing $A = A^{\mathbf{K}} \otimes s_{\mathbf{K}}$ we have

$$D_{e_\mu} A = \partial_\mu A^{\mathbf{K}} \otimes s_{\mathbf{K}} + A^{\mathbf{K}} \otimes D_{e_\mu} s_{\mathbf{K}}. \quad (122)$$

Now, let examine the case where $E(M) = TM \equiv \bigwedge^1(TM) \hookrightarrow \mathcal{C}\ell(TM)$. Let $\{\mathbf{e}_j\}$, be an orthonormal basis of TM . Then, using Eq.(122)

$$\begin{aligned} \mathbf{d}^E \mathbf{e}_j &= (D_{\mathbf{e}_k} \mathbf{e}_j) \otimes \theta^k \equiv \mathbf{e}_k \otimes \omega_j^k \\ \omega_j^k &= \omega_{rj}^k \theta^r, \end{aligned} \quad (123)$$

where the $\omega_j^k \in \sec \bigwedge^1 T^*M$ are the so-called *connection 1-forms*.

Also, for $\mathbf{v} = v^i \mathbf{e}_i \in \sec TM$, we have

$$\begin{aligned} \mathbf{d}^E \mathbf{v} &= D_{\mathbf{e}_i} \mathbf{v} \otimes \theta^i = \mathbf{e}_i \otimes \mathbf{d}^E v^i, \\ \mathbf{d}^E v^i &= dv^i + \omega_k^i v^k. \end{aligned} \quad (124)$$

4.3 Absolute Differential

Now, let $E(M) = TM \equiv \bigwedge^l(TM) \hookrightarrow \mathcal{C}\ell(TM)$. Recall that the usual *absolute differential* D of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ is a mapping (see, e.g., [26])

$$D: \sec \bigwedge^l TM \rightarrow \sec \bigwedge^l TM \otimes \bigwedge^1 T^*M, \quad (125)$$

such that for any differentiable $A \in \sec \bigwedge^l TM$ we have

$$DA = (D_{\mathbf{e}_i} A) \otimes \theta^i, \quad (126)$$

where $D_{\mathbf{e}_i} A$ is the standard covariant derivative of $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$.

Also, for any differentiable function $f: M \rightarrow \mathbb{R}$, and differentiable $A \in \sec \bigwedge^l TM$ we have

$$D(fA) = df \otimes A + fDA. \quad (127)$$

Now, if we suppose that the orthonormal basis $\{\mathbf{e}_j\}$ of TM is such that each $\mathbf{e}_j \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$, we can find easily using the Clifford algebra structure of the space of multivectors that Eq.(123) can be written as:

$$\begin{aligned} D\mathbf{e}_j &= (D_{\mathbf{e}_k} \mathbf{e}_j) \theta^k = \frac{1}{2} [\omega, \mathbf{e}_j] = -\mathbf{e}_j \lrcorner \omega \\ \omega &= \frac{1}{2} \omega_{\mathbf{k}}^{\mathbf{ab}} \mathbf{e}_a \wedge \mathbf{e}_b \otimes \theta^k \\ &\equiv \frac{1}{2} \omega_{\mathbf{k}}^{\mathbf{ab}} \mathbf{e}_a \mathbf{e}_b \otimes \theta^k \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M, \end{aligned} \quad (128)$$

where ω is the *representative* of the connection in a given gauge.

The general case is given by the following proposition.

Proposition 3 For $A \in \sec \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$

$$DA = dA + \frac{1}{2}[\omega, A]. \quad (129)$$

Proof. The proof is a simple calculation, left to the reader. ■

Eq.(129) can now be extended by linearity for an arbitrary nonhomogeneous multivector $A \in \sec \mathcal{C}\ell(TM)$.

Remark 4 We see that when $E(M) = \bigwedge^l TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ the absolute differential D can be identified with the exterior covariant derivative \mathbf{d}^E .

We proceed now to find an appropriate *exterior* covariant differential which acts naturally on Clifford valued differential forms, i.e., objects that are sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$ ($\equiv \bigwedge T^*M \otimes \mathcal{C}\ell(TM)$) (see next section). Note that we cannot simply use the above definition by using $E(M) = \mathcal{C}\ell(TM)$ and $\text{end}E(M) = \text{end}\mathcal{C}\ell(TM)$, because $\text{end}\mathcal{C}\ell(TM) \neq \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. Instead, we must use the above theory and possible applications as a guide in order to find an appropriate definition. Let us see how this can be done.

5 Clifford Valued Differential Forms

Definition 5 A homogeneous multivector valued differential form of type (l, p) is a section of $\bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge T^*M$, for $0 \leq l \leq 4$, $0 \leq p \leq 4$. A section of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$ such that the multivector part is non homogeneous is called a Clifford valued differential form.

We recall, that any $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ can always be written as

$$\begin{aligned} A &= m_{(l)} \otimes \psi^{(p)} \equiv \frac{1}{l!} m_{(l)}^{i_1 \dots i_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_l} \otimes \psi^{(p)} \\ &= \frac{1}{p!} m_{(l)} \otimes \psi_{j_1 \dots j_p}^{(p)} \theta^{j_1} \wedge \dots \wedge \theta^{j_p} \\ &= \frac{1}{l!p!} m_{(l)}^{i_1 \dots i_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_l} \otimes \psi_{j_1 \dots j_p}^{(p)} \theta^{j_1} \wedge \dots \wedge \theta^{j_p} \\ &= \frac{1}{l!p!} A_{j_1 \dots j_p}^{i_1 \dots i_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_l} \otimes \theta^{i_1} \wedge \dots \wedge \theta^{i_p}. \end{aligned} \quad (130)$$

Definition 6 The \otimes_\wedge product of $A = \overset{m}{A} \otimes \psi^{(p)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ and $B = \overset{m}{B} \otimes \chi^{(q)} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$ is the mapping:

$$\begin{aligned} \otimes_\wedge : \sec \mathcal{C}\ell(TM) \otimes \bigwedge^l T^*M &\times \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M \\ &\rightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^{l+p} T^*M, \\ A \otimes_\wedge B &= \overset{m}{A} \overset{m}{B} \otimes \psi^{(p)} \wedge \chi^{(q)}. \end{aligned} \quad (131)$$

Definition 7 The commutator $[A, B]$ of $A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ and $B \in \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$ is the mapping:

$$\begin{aligned} [\quad , \quad] : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M &\times \sec \bigwedge^m TM \otimes \bigwedge^q T^*M \\ &\rightarrow \sec \left(\left(\sum_{k=|l-m|}^{|l+m|} \bigwedge^k T^*M \right) \otimes \bigwedge^{p+q} T^*M \right) \\ [A, B] &= A \otimes_\wedge B - (-1)^{pq} B \otimes_\wedge A. \end{aligned} \quad (132)$$

Writing $A = \frac{1}{l!} A^{j_1 \dots j_l} \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \psi^{(p)}$, $B = \frac{1}{m!} B^{i_1 \dots i_m} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m} \chi^{(q)}$, with $\psi^{(p)} \in \sec \bigwedge^p T^*M$ and $\chi^{(q)} \in \sec \bigwedge^q T^*M$, we have

$$[A, B] = \frac{1}{l!m!} A^{j_1 \dots j_l} B^{i_1 \dots i_m} [\mathbf{e}_{j_1} \dots \mathbf{e}_{j_l}, \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m}] \psi^{(p)} \wedge \chi^{(q)}, \quad (133)$$

The definition of the commutator is extended by linearity to arbitrary sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$.

Now, we have the proposition.

Proposition 8 Let $A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, $B \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$, $C \in A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^r T^*M$. Then,

$$[A, B] = (-1)^{1+pq} [B, A], \quad (134)$$

and

$$(-1)^{pr} [[A, B], C] + (-1)^{qp} [[B, C], A] + (-1)^{rq} [[C, A], B] = 0. \quad (135)$$

Proof. It follows directly from a simple calculation, left to the reader. ■

Eq.(135) may be called the *graded Jacobi identity* [20].

Corollary 9 Let $A^{(2)} \in \sec \bigwedge^2(TM) \otimes \bigwedge^p T^*M$ and $B \in \sec \bigwedge^r(TM) \otimes \bigwedge^q T^*M$. Then,

$$[A^{(2)}, B] = C, \quad (136)$$

where $C \in \sec \bigwedge^r(TM) \otimes \bigwedge^{p+q} T^*M$.

Proof. It follows from a direct calculation, left to the reader. ■

Proposition 10 Let $\omega \in \sec \bigwedge^2(TM) \otimes \bigwedge^1 T^*M$, $A \in \sec \bigwedge^l(TM) \otimes \bigwedge^p T^*M$, $B \in \sec \bigwedge^m(TM) \otimes \bigwedge^q T^*M$. Then, we have

$$(p+q)[\omega, A \otimes_\wedge B] = p[\omega, A] \otimes_\wedge B + (-1)^p q A \otimes_\wedge [\omega, B]. \quad (137)$$

Proof. Write, $\omega = \frac{1}{2} \omega_i^{ab} \mathbf{e}_a \mathbf{e}_b \theta^i$, $A = \frac{1}{m!} A^{j_1 \dots j_l} \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} A^{(p)}$, $B = \frac{1}{m!} B^{i_1 \dots i_m} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m} B^{(q)}$. Then,

$$\begin{aligned} (p+q)[\omega, A \otimes_\wedge B] &= (p+q) \frac{1}{2l!m!} \omega_i^{ab} A^{j_1 \dots j_l} B^{i_1 \dots i_m} [\mathbf{e}_a \mathbf{e}_b, \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m}] \otimes \theta^i \wedge A^{(p)} \wedge B^{(q)} \\ &= p \frac{1}{2l!m!} \omega_i^{ab} A^{j_1 \dots j_l} B^{i_1 \dots i_m} [\mathbf{e}_a \mathbf{e}_b, \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m}] \otimes \theta^i \wedge A^{(p)} \wedge B^{(q)} \\ &\quad + q \frac{1}{2l!m!} \omega_i^{ab} A^{j_1 \dots j_l} B^{i_1 \dots i_m} [\mathbf{e}_a \mathbf{e}_b, \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \mathbf{e}_{i_1} \dots \mathbf{e}_{i_m}] \otimes \theta^i \wedge A^{(p)} \wedge B^{(q)} \\ &= pA[\omega, A] \otimes_\wedge B + (-1)^p q A \otimes_\wedge [\omega, B]. \end{aligned}$$

■

Definition 11 The action of the differential operator d acting on

$$A \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M,$$

is given by:

$$\begin{aligned} dA &\stackrel{\circ}{=} \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \otimes dA^{j_1 \dots j_l} \\ &= \mathbf{e}_{j_1} \dots \mathbf{e}_{j_l} \otimes d \frac{1}{p!} A_{i_1 \dots i_p}^{j_1 \dots j_l} \theta^{i_1} \wedge \dots \wedge \theta^{i_p}. \end{aligned} \quad (138)$$

We have the important proposition.

Proposition 12 Let $A \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$ and $B \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$. Then,

$$d[A, B] = [dA, B] + (-1)^p [A, dB]. \quad (139)$$

Proof. The proof of that proposition is a simple calculation, left to the reader.

■

We now define the exterior covariant differential operator (*EXCD*) \mathbf{D} and the *extended* covariant derivative (*ECD*) $\mathbf{D}_{\mathbf{e}_r}$ acting on a Clifford valued form $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, as follows.

5.1 Exterior Covariant Differential of Clifford Valued Forms

Definition 13 *The exterior covariant differential of \mathcal{A} is the mapping :*

$$\begin{aligned} \mathbf{D}: \sec \bigwedge^l TM \otimes \bigwedge^p T^*M &\rightarrow \sec[(\bigwedge^l TM \otimes \bigwedge^p T^*M) \otimes_{\wedge} \bigwedge^1 T^*M] \\ &\subset \sec \bigwedge^l TM \otimes \bigwedge^{p+1} T^*M, \\ \mathbf{D}\mathcal{A} &= d\mathcal{A} + \frac{p}{2}[\omega, \mathcal{A}], \text{ if } \mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M, \ l, p \geq 1. \end{aligned} \quad (140)$$

Proposition 14 *Let be $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, $\mathcal{B} \in \sec \bigwedge^m TM \otimes \bigwedge^q T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^q T^*M$. Then, the exterior differential satisfies*

$$\mathbf{D}(\mathcal{A} \otimes_{\wedge} \mathcal{B}) = \mathbf{D}\mathcal{A} \otimes_{\wedge} \mathcal{B} + (-1)^p \mathcal{A} \otimes_{\wedge} \mathbf{D}\mathcal{B} \quad (141)$$

Proof. It follows directly from the definition if we take into account the properties of the product \otimes_{\wedge} and Eq.(137). ■

5.2 Extended Covariant Derivative of Clifford Valued Forms

Definition 15 *The extended covariant derivative operator is the mapping*

$$\mathbf{D}_{e_r} : \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \rightarrow \sec \bigwedge^l TM \otimes \bigwedge^p T^*M,$$

*such that for any $\mathcal{A} \in \sec \bigwedge^l TM \otimes \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^p T^*M$, $l, p \geq 1$, we have*

$$\mathbf{D}\mathcal{A} = (\mathbf{D}_{e_r}\mathcal{A}) \otimes_{\wedge} \theta^r. \quad (142)$$

We can immediately verify that

$$\mathbf{D}_{e_r}\mathcal{A} = \partial_{e_r}\mathcal{A} + \frac{p}{2}[\omega_r, \mathcal{A}], \quad (143)$$

and, of course, in general³⁵

$$\mathbf{D}_{e_r}\mathcal{A} \neq D_{e_r}\mathcal{A} \quad (144)$$

Let us write explicitly some important cases which will appear latter.

³⁵For a Clifford algebra formula for the calculation of $D_{e_r}\mathcal{A}$, $\mathcal{A} \in \sec \bigwedge^p T^*M$ see Eq.(255).

5.2.1 Case $p = 1$

Let $\mathcal{A} \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$. Then,

$$\mathbf{D}\mathcal{A} = d\mathcal{A} + \frac{1}{2}[\omega, \mathcal{A}], \quad (145)$$

and

$$\mathbf{D}_{e_k}\mathcal{A} = \partial_{e_k}\mathcal{A} + \frac{1}{2}[\omega_k, \mathcal{A}]. \quad (146)$$

5.2.2 Case $p = 2$

Let $\mathcal{F} \in \sec \bigwedge^2 TM \otimes \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M$. Then,

$$\mathbf{D}\mathcal{F} = d\mathcal{F} + [\omega, \mathcal{F}], \quad (147)$$

and

$$\mathbf{D}_{e_r}\mathcal{F} = \partial_{e_r}\mathcal{F} + [\omega_r, \mathcal{F}]. \quad (148)$$

5.3 Cartan Exterior Differential

Recall that [50] *Cartan* defined the exterior covariant differential of $\mathfrak{C} = e_i \otimes \mathfrak{C}^i \in \sec \bigwedge^1 TM \otimes \bigwedge^p T^*M$ as a mapping

$$\begin{aligned} \mathbf{D}^c: \bigwedge^1 TM \otimes \bigwedge^p T^*M &\longrightarrow \bigwedge^1 TM \otimes \bigwedge^{p+1} T^*M, \\ \mathbf{D}^c \mathfrak{C} = \mathbf{D}^c(e_i \otimes \mathfrak{C}^i) &= e_i \otimes d\mathfrak{C}^i + \mathbf{D}^c e_i \wedge \mathfrak{C}^i, \\ \mathbf{D}^c e_j &= (D_{e_k} e_j) \theta^k \end{aligned} \quad (149)$$

which in view of Eq.(138) and Eq.(128) can be written as

$$\mathbf{D}^c \mathfrak{C} = \mathbf{D}^c(e_i \otimes \mathfrak{C}^i) = d\mathfrak{C} + \frac{1}{2}[\omega, \mathfrak{C}]. \quad (150)$$

So, we have, for $p > 1$, the following relation between the exterior covariant differential \mathbf{D} and Cartan's exterior differential ($p > 1$)

$$\mathbf{D}\mathfrak{C} = \mathbf{D}^c \mathfrak{C} + \frac{p-1}{2}[\omega, \mathfrak{C}]. \quad (151)$$

Note moreover that when $\mathfrak{C}^{(1)} = e_i \otimes \mathfrak{C}^i \in \sec \bigwedge^1 TM \otimes \bigwedge^1 T^*M$, we have

$$\mathbf{D}\mathfrak{C}^{(1)} = \mathbf{D}^c \mathfrak{C}^{(1)}. \quad (152)$$

We end this section with two observations:

(i) There are other approaches to the concept of exterior covariant differential acting on sections of a vector bundle $E \otimes \bigwedge^p T^*M$ and also in sections of $\text{end}(E) \otimes \bigwedge^p T^*M$, as e.g., in [18, 19, 50, 52, 72, 74, 98]. Not all are completely equivalent among themselves and to the one presented above. Our definitions, we think, have the merit of mimicking coherently the pullback under a local section of the covariant differential acting on sections of vector bundles associated to a given principal bundle as used in gauge theories. Indeed, this consistence will be checked in several situations below.

(ii) Some authors, e.g., [19, 101] find convenient to introduce the concept of *exterior covariant derivative* of indexed p -forms, which are objects like the curvature 2-forms (see below) or the connection 1-forms introduced above. We do not use such concept in this paper.

5.4 Torsion and Curvature

Let $\theta = e_\mu dx^\mu = \mathbf{e}_a \theta^a \in \text{sec} \bigwedge^1 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ and $\omega = \frac{1}{2} (\omega_a^{bc} \mathbf{e}_b \wedge \mathbf{e}_c) \otimes \theta^a \equiv \frac{1}{2} \omega_a^{bc} \mathbf{e}_b \mathbf{e}_c \theta^a \in \text{sec} \bigwedge^2 M \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ be respectively the *representatives* of a soldering form and a connection on the *basis manifold*. Then, following the standard procedure [57], the *torsion* of the connection and the *curvature* of the connection on the basis manifold are defined by

$$\Theta = D\theta \in \text{sec} \bigwedge^1 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M, \quad (153)$$

and

$$\mathcal{R} = D\omega \in \text{sec} \bigwedge^2 M \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M. \quad (154)$$

We now calculate Θ and $D\mathcal{R}$. We have,

$$D\theta = D(\mathbf{e}_a \theta^a) = \mathbf{e}_a d\theta^a + \frac{1}{2} [\omega_a, \mathbf{e}_d] \theta^a \wedge \theta^d \quad (155)$$

and since $\frac{1}{2} [\omega_a, \mathbf{e}_d] = -\mathbf{e}_d \lrcorner \omega_a = \omega_{ad}^c \mathbf{e}_c$ we have

$$D(\mathbf{e}_a \theta^a) = \mathbf{e}_a [d\theta^a + \omega_{bd}^a \theta^b \wedge \theta^d] = \mathbf{e}_a \Theta^a, \quad (156)$$

and we recognize

$$\Theta^a = d\theta^a + \omega_{bd}^a \theta^b \wedge \theta^d, \quad (157)$$

as *Cartan's first structure equation*.

For a torsion free connection, the torsion 2-forms $\Theta^a = 0$, and it follows that $\Theta = 0$. A metrical compatible connection ($Dg = 0$) satisfying $\Theta^a = 0$ is called a Levi-Civita connection. In the remaining of this paper we *restrict* ourself to that case.

Now, according to Eq.(140) we have,

$$D\mathcal{R} = d\mathcal{R} + [\omega, \mathcal{R}]. \quad (158)$$

Now, taking into account that

$$\mathcal{R} = d\omega + \frac{1}{2}[\omega, \omega], \quad (159)$$

and that from Eqs.(134).(135) and (139) it follows that

$$\begin{aligned} d[\omega, \omega] &= [d\omega, \omega] - [\omega, d\omega], \\ [d\omega, \omega] &= -[\omega, d\omega], \\ [[\omega, \omega], \omega] &= 0, \end{aligned} \quad (160)$$

we have immediately

$$\mathbf{D}\mathcal{R} = d\mathcal{R} + [\omega, \mathcal{R}] = 0. \quad (161)$$

Eq.(161) is known as the *Bianchi identity*.

Note that

$$\begin{aligned} \mathcal{R} &= \frac{1}{4}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}} \wedge \mathbf{e}_{\mathbf{b}} \otimes (dx^\mu \wedge dx^\nu) \\ &\equiv \frac{1}{4}\mathcal{R}_{\mathbf{cd}}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} \otimes \theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}} = \frac{1}{4}R_{\rho\sigma}^{\alpha\beta}e_\alpha e_\beta \otimes dx^\rho \wedge dx^\sigma \\ &= \frac{1}{4}R_{\mu\nu\rho\sigma}e^\mu e^\nu \otimes dx^\rho \wedge dx^\sigma, \end{aligned} \quad (162)$$

where $R_{\mu\nu\rho\sigma}$ are the components of the curvature tensor, also known in differential geometry as the Riemann tensor. We recall the well known symmetries

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= -R_{\nu\mu\rho\sigma}, \\ R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho}, \\ R_{\mu\nu\rho\sigma} &= R_{\rho\sigma\mu\nu}. \end{aligned} \quad (163)$$

We also write Eq.(162) as

$$\begin{aligned} \mathcal{R} &= \frac{1}{4}R_{\mathbf{cd}}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} \otimes (\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) = \frac{1}{2}\mathbf{R}_{\mu\nu}dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}}, \end{aligned} \quad (164)$$

with

$$\begin{aligned} \mathbf{R}_{\mu\nu} &= \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}} \wedge \mathbf{e}_{\mathbf{b}} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM), \\ \mathcal{R}^{\mathbf{ab}} &= \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}dx^\mu \wedge dx^\nu \in \sec \bigwedge^2 T^*M, \end{aligned} \quad (165)$$

where $\mathbf{R}_{\mu\nu}$ will be called curvature bivectors and the $\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}$ are called after Cartan the curvature 2-forms. The $\mathcal{R}_{\mathbf{b}}^{\mathbf{a}}$ satisfy *Cartan's second structure equation*

$$\mathcal{R}_{\mathbf{b}}^{\mathbf{a}} = d\omega_{\mathbf{b}}^{\mathbf{a}} + \omega_{\mathbf{c}}^{\mathbf{a}} \wedge \omega_{\mathbf{d}}^{\mathbf{c}}, \quad (166)$$

which follows calculating $d\mathcal{R}$ from Eq.(159). Now, we can also write,

$$\begin{aligned}
D\mathcal{R} &= d\mathcal{R} + [\omega, \mathcal{R}] \\
&= \frac{1}{2} \left\{ d\left(\frac{1}{2} R_{\mu\nu}^{\mathbf{ab}} \mathbf{e}_{\mathbf{a}} \mathbf{e}_{\mathbf{b}} dx^\mu \wedge dx^\nu\right) + \frac{1}{2} [\omega_\rho, \mathbf{R}_{\mu\nu}] \right\} dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{2} \left\{ \partial_\rho \mathbf{R}_{\mu\nu} + [\omega_\rho, \mathbf{R}_{\mu\nu}] \right\} dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{2} \mathbf{D}_{e_\rho} \mathbf{R}_{\mu\nu} dx^\rho \wedge dx^\mu \wedge dx^\nu \\
&= \frac{1}{3!} (\mathbf{D}_{e_\rho} \mathbf{R}_{\mu\nu} + \mathbf{D}_{e_\mu} \mathbf{R}_{\nu\rho} + \mathbf{D}_{e_\nu} \mathbf{R}_{\rho\mu}) dx^\rho \wedge dx^\mu \wedge dx^\nu = 0,
\end{aligned} \tag{167}$$

from where it follows that

$$\mathbf{D}_{e_\rho} \mathbf{R}_{\mu\nu} + \mathbf{D}_{e_\mu} \mathbf{R}_{\nu\rho} + \mathbf{D}_{e_\nu} \mathbf{R}_{\rho\mu} = 0. \tag{168}$$

Remark 16 Eq.(168) is called in Physics textbooks on gauge theories (see, e.g., [72, 89]) *Bianchi identity*. Note that physicists call the extended covariant derivative operator

$$\mathbf{D}_{e_\rho} \equiv \mathbf{D}_\rho = \partial_\rho + [\omega_\rho,], \tag{169}$$

acting on the curvature bivectors as the ‘covariant derivative’. Note however that, as detailed above, this operator is not the usual covariant derivative operator $D_{\mathbf{e}_a}$ acting on sections of the tensor bundle.

We now find the explicit expression for the curvature bivectors $\mathbf{R}_{\mu\nu}$ in terms of the connections bivectors $\omega_\mu = \omega(e_\mu)$, which will be used latter. First recall that by definition

$$\mathbf{R}_{\mu\nu} = \mathcal{R}(e_\mu, e_\nu) = -\mathcal{R}(e_\nu, e_\mu) = -\mathbf{R}_{\nu\mu}. \tag{170}$$

Now, observe that using Eqs.(134), (135) and (139) we can easily show that

$$\begin{aligned}
[\omega, \omega](e_\mu, e_\nu) &= 2[\omega(e_\mu), \omega(e_\nu)] \\
&= 2[\omega_\mu, \omega_\nu].
\end{aligned} \tag{171}$$

Using Eqs. (159), (170) and (171) we get

$$\mathbf{R}_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]. \tag{172}$$

5.5 Some Useful Formulas

Proposition 17 Let $A \in \sec \bigwedge^p TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ and \mathcal{R} the curvature of the connection as defined in Eq.(154). Then,

$$D^2 A = \frac{1}{2} [\mathcal{R}, A]. \tag{173}$$

Proof. The first member is

$$\begin{aligned}\mathbf{D}^2 A &= \mathbf{D} \mathbf{D} A = \mathbf{D} \left(dA + \frac{1}{2} [\boldsymbol{\omega}, A] \right) \\ &= d^2 A + \frac{1}{2} [\boldsymbol{\omega}, dA] + \frac{1}{2} d[\boldsymbol{\omega}, A] + \frac{1}{4} [\boldsymbol{\omega}, [\boldsymbol{\omega}, A]].\end{aligned}\quad (174)$$

Now, as can be easily verified,

$$d[\boldsymbol{\omega}, A] = [d\boldsymbol{\omega}, A] - [\boldsymbol{\omega}, dA], \quad (175)$$

$$[\boldsymbol{\omega}, [\boldsymbol{\omega}, A]] = [[\boldsymbol{\omega}, \boldsymbol{\omega}], A], \quad (176)$$

$$\frac{1}{4} [\boldsymbol{\omega}, [\boldsymbol{\omega}, A]] = \frac{1}{2} [\boldsymbol{\omega} \otimes_{\wedge} \boldsymbol{\omega}, A]. \quad (177)$$

Using these equations in Eq.(174) we have,

$$\mathbf{D}^2 A = \frac{1}{2} [d\boldsymbol{\omega} + \boldsymbol{\omega} \otimes_{\wedge} \boldsymbol{\omega}, A] = \frac{1}{2} [\mathcal{R}, A].$$

■

In particular, when $a \in \sec \bigwedge^1 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$ we have

$$\mathbf{D}^2 a = \mathcal{R} \lrcorner a \quad (178)$$

Also, we can show using the previous result that if $\mathcal{A} \in \sec \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ it holds

$$\mathbf{D}^2 \mathcal{A} = \frac{1}{2} [\mathcal{R}, \mathcal{A}]. \quad (179)$$

It is a useful test of the consistence of our formalism to derive once again that $\mathbf{D}\mathcal{R} = 0$, by calculating $\mathbf{D}^3 A$ for $A \in \sec \bigwedge^r TM \hookrightarrow \sec \mathcal{C}\ell(TM)$. We have:

$$\mathbf{D}^3 A = \mathbf{D} (\mathbf{D}^2 A) = \mathbf{D}^2 (\mathbf{D} A). \quad (180)$$

Now, using the above formulas and recalling Eq.(173), we can write:

$$\begin{aligned}\mathbf{D}^3 A &= \mathbf{D} (\mathbf{D}^2 A) = \frac{1}{2} \mathbf{D} [\mathcal{R}, A] \\ &= \frac{1}{2} \mathbf{D} (\mathcal{R} \otimes_{\wedge} A - A \otimes_{\wedge} \mathcal{R}) \\ &= \frac{1}{2} (\mathbf{D}\mathcal{R} \otimes_{\wedge} A + \mathcal{R} \otimes_{\wedge} \mathbf{D} A - \mathbf{D} A \otimes_{\wedge} \mathcal{R} + (-1)^{1+r} A \otimes_{\wedge} \mathbf{D}\mathcal{R})\end{aligned}\quad (181)$$

and

$$\begin{aligned}\mathbf{D}^3 A &= \mathbf{D}^2 (\mathbf{D} A) = \frac{1}{2} [\mathcal{R}, \mathbf{D} A] \\ &= \frac{1}{2} (\mathcal{R} \otimes_{\wedge} \mathbf{D} A - \mathbf{D} A \otimes_{\wedge} \mathcal{R}).\end{aligned}\quad (182)$$

Comparing Eqs.(181) and (182) we get that

$$\mathbf{D}\mathcal{R} \otimes_{\wedge} A + (-1)^{1+r} A \otimes_{\wedge} \mathbf{D}\mathcal{R} = [\mathbf{D}\mathcal{R}, A] = 0, \quad (183)$$

from where it follows that $\mathbf{D}\mathcal{R} = 0$, as it may be.

6 General Relativity as a $Sl(2, \mathbb{C})$ Gauge Theory

6.1 The Nonhomogeneous Field Equations

The analogy of the fields $\mathbf{R}_{\mu\nu} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} = \frac{1}{2}R_{\mu\nu}^{\mathbf{ab}}\mathbf{e}_{\mathbf{a}} \wedge \mathbf{e}_{\mathbf{b}} \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM)$ with the gauge fields of particle fields is so appealing that it is irresistible to propose some kind of a $Sl(2, \mathbb{C})$ formulation for the gravitational field. And indeed this has already been done, and the interested reader may consult, e.g., [22, 65]. Here, we observe that despite the similarities, the gauge theories of particle physics are in general formulated in flat Minkowski spacetime and the theory here must be for a field on a general Lorentzian spacetime. This introduces additional complications, but it is not our purpose to discuss that issue with all attention it deserves here. Indeed, for our purposes in this paper we will need only to recall some facts.

To start, recall that in gauge theories besides the homogenous field equations given by Bianchi's identities, we also have the nonhomogeneous field equation. This equation, in analogy to the nonhomogeneous equation for the electromagnetic field (see Eq.(264) in Appendix A) is written here as

$$\mathbf{D}\star\mathcal{R} = d\star\mathcal{R} + \frac{1}{2}[\omega, \star\mathcal{R}] = -\star\mathcal{J}, \quad (184)$$

where the $\mathcal{J} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M$ is a ‘current’, which, if the theory is to be one equivalent to General Relativity, must be in some way related with the energy momentum tensor in Einstein theory. In order to write from this equation an equation for the curvature bivectors, it is very useful to imagine that $\bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M)$, the Clifford bundle of differential forms, for in that case the powerful calculus described in the Appendix A can be used. So, we write:

$$\begin{aligned} \omega &\in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M), \\ \mathcal{R} = \mathbf{D}\omega &\in \sec \bigwedge^2 TM \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \bigwedge^2 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M) \\ \mathcal{J} = \mathbf{J}_{\nu} \otimes \theta^{\nu} &\equiv \mathbf{J}_{\nu} \theta^{\nu} \in \sec \bigwedge^2 TM \otimes \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(TM) \otimes \mathcal{C}\ell(T^*M). \end{aligned} \quad (185)$$

Now, using Eq.(253) for the Hodge star operator given in the Appendix A.3 and the relation between the operators $d = \boldsymbol{\partial} \wedge$ and $\delta = -\boldsymbol{\partial} \lrcorner$ (Appendix A5) we can write

$$d\star\mathcal{R} = -\theta^5(-\boldsymbol{\partial} \lrcorner \mathcal{R}) = -\star(\boldsymbol{\partial} \lrcorner \mathcal{R}) = -\star((\partial_{\mu}\mathbf{R}_{\nu}^{\mu})\theta^{\nu}). \quad (186)$$

Also,

$$\begin{aligned}
\frac{1}{2}[\omega, \star \mathcal{R}] &= \frac{1}{2}[\omega_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \star(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{2}[\omega_\mu, \mathbf{R}_{\alpha\beta}] \otimes \theta^\mu \wedge \theta^5(\theta^\alpha \wedge \theta^\beta) \\
&= -\frac{1}{4}[\omega_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \theta^5(\theta^\alpha \wedge \theta^\beta) + \theta^5(\theta^\alpha \wedge \theta^\beta) \theta^\mu\} \\
&= \frac{\theta^5}{4}[\omega_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu(\theta^\alpha \wedge \theta^\beta) - (\theta^\alpha \wedge \theta^\beta) \theta^\mu\} \\
&= \frac{\theta^5}{2}[\omega_\mu, \mathbf{R}_{\alpha\beta}] \otimes \{\theta^\mu \lrcorner (\theta^\alpha \wedge \theta^\beta)\} \\
&= -\star([\omega_\mu, \mathbf{R}_{\alpha\beta}^\mu] \theta^\beta).
\end{aligned} \tag{187}$$

Using Eqs.(184-187) we get³⁶

$$\partial_\mu \mathbf{R}_\nu^\mu + [\omega_\mu, \mathbf{R}_\nu^\mu] = \mathbf{D}_{e_\mu} \mathbf{R}_\nu^\mu = \mathbf{J}_\nu. \tag{188}$$

So, the gauge theory of gravitation has as field equations the Eq.(188), the nonhomogeneous field equations, and Eq. (168) the homogeneous field equations (which is Bianchi's identity). We summarize that equations, as

$$\mathbf{D}_{e_\mu} \mathbf{R}_\nu^\mu = \mathbf{J}_\nu, \quad \mathbf{D}_{e_\rho} \mathbf{R}_{\mu\nu} + \mathbf{D}_{e_\mu} \mathbf{R}_{\nu\rho} + \mathbf{D}_{e_\nu} \mathbf{R}_{\rho\mu} = 0. \tag{189}$$

Eqs.(189) which looks like Maxwell equations, must, of course, be compatible with Einstein's equations, which may be eventually used to determine determines \mathbf{R}_ν^μ , ω_μ and \mathbf{J}_ν .

7 Another Set of Maxwell Like Nonhomogeneous Equations for Einstein Theory

We now show, e.g., how a special combination of the $\mathbf{R}_\mathbf{b}^\mathbf{a}$ are directly related with a combination of products of the energy-momentum 1-vectors $T_\mathbf{a}$ and the tetrad fields $\mathbf{e}_\mathbf{a}$ (see Eq.(192) below) in Einstein theory. In order to do that, we recall that Einstein's equations can be written in components in an orthonormal basis as

$$R_{\mathbf{ab}} - \frac{1}{2}\eta_{\mathbf{ab}}R = T_{\mathbf{ab}}, \tag{190}$$

where $R_{\mathbf{ab}} = R_{\mathbf{ba}}$ are the components of the Ricci tensor ($R_{\mathbf{ab}} = R_{\mathbf{a}^\mathbf{c} \mathbf{b} \mathbf{c}}$), $T_{\mathbf{ab}}$ are the components of the energy-momentum tensor of matter fields and $R = \eta_{\mathbf{ab}} R^{\mathbf{ab}}$ is the curvature scalar. We next introduce³⁷ the *Ricci 1-vectors*

³⁶Recall that $\mathbf{J}_\nu \in \sec \wedge^2 TM \hookrightarrow \sec \mathcal{C}\ell(TM)$.

³⁷Ricci 1-form fields appear naturally when we formulate Einstein's equations in terms of tetrad fields. See Appendix B.

and the *energy-momentum 1-vectors* by

$$R_{\mathbf{a}} = R_{\mathbf{ab}}\mathbf{e}^{\mathbf{b}} \in \sec \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(TM), \quad (191)$$

$$T_{\mathbf{a}} = T_{\mathbf{ab}}\mathbf{e}^{\mathbf{b}} \in \sec \bigwedge^1 TM \hookrightarrow \mathcal{C}\ell(TM). \quad (192)$$

We have that

$$R_{\mathbf{a}} = -\mathbf{e}^{\mathbf{b}} \lrcorner \mathbf{R}_{\mathbf{ab}}. \quad (193)$$

Now, multiplying Eq.(190) on the right by $\mathbf{e}^{\mathbf{b}}$ we get

$$R_{\mathbf{a}} - \frac{1}{2}R\mathbf{e}_{\mathbf{a}} = T_{\mathbf{a}}. \quad (194)$$

Multiplying Eq.(194) first on the right by $\mathbf{e}_{\mathbf{b}}$ and then on the left by $\mathbf{e}_{\mathbf{b}}$ and making the difference of the resulting equations we get

$$(-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}(-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}) = (T_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}T_{\mathbf{a}}). \quad (195)$$

Defining

$$\begin{aligned} \mathcal{F}_{\mathbf{ab}} &= (-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}})\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}(-\mathbf{e}^{\mathbf{c}} \lrcorner \mathbf{R}_{\mathbf{ac}}) - \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}) \\ &= \frac{1}{2}(R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{b}}\mathbf{e}^{\mathbf{c}}R_{\mathbf{ac}} - \mathbf{e}^{\mathbf{c}}R_{\mathbf{ac}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}R_{\mathbf{ac}}\mathbf{e}^{\mathbf{c}}) - \frac{1}{2}R(\mathbf{e}_{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}\mathbf{e}_{\mathbf{a}}) \end{aligned} \quad (196)$$

and

$$\mathcal{J}_{\mathbf{b}} = D_{\mathbf{e}_{\mathbf{a}}}(T^{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}T^{\mathbf{a}}), \quad (197)$$

we have³⁸

$$D_{\mathbf{e}_{\mathbf{a}}}\mathcal{F}_{\mathbf{b}}^{\mathbf{a}} = \mathcal{J}_{\mathbf{b}} \quad (198)$$

It is quite obvious that in a coordinate chart $\langle x^{\mu} \rangle$ covering an open set $U \subset M$ we can write

$$D_{e_{\rho}}\mathcal{F}_{\beta}^{\rho} = \mathcal{J}_{\beta}, \quad (199)$$

with $\mathcal{F}_{\beta}^{\rho} = g^{\rho\alpha}\mathcal{F}_{\alpha\beta}$

$$\mathcal{F}_{\alpha\beta} = (-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma})e_{\beta} - e_{\beta}(-e^{\gamma} \lrcorner \mathbf{R}_{\alpha\gamma}) - \frac{1}{2}R(e_{\alpha}e_{\beta} - e_{\beta}e_{\alpha}) \quad (200)$$

$$\mathcal{J}_{\beta} = D_{e_{\rho}}(T^{\rho}e_{\beta} - e^{\rho}T_{\beta}). \quad (201)$$

³⁸Note that we could also produce another Maxwell like equation, by using the extended covariant derivative operator in the definition of the current, i.e., we can put $\mathcal{J}_{\mathbf{b}} = D_{\mathbf{e}_{\mathbf{a}}}(T^{\mathbf{a}}\mathbf{e}_{\mathbf{b}} - \mathbf{e}_{\mathbf{b}}T^{\mathbf{a}})$, and in that case we obtain $D_{\mathbf{e}_{\mathbf{a}}}\mathcal{F}_{\mathbf{b}}^{\mathbf{a}} = \mathcal{J}_{\mathbf{b}}$.

Remark 18 *Eq.(198) (or Eq.(199)) is a set Maxwell like nonhomogeneous equations. It looks like the nonhomogeneous classical Maxwell equations when that equations are written in components, but Eq.(199) is only a new way of writing the equation of the nonhomogeneous field equations in the $Sl(2, \mathbb{C})$ like gauge theory version of Einstein's theory, discussed in the previous section. In particular, recall that any one of the six $\mathcal{F}_\beta^\rho \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{Cl}(TM)$. Or, in words, each one of the \mathcal{F}_β^ρ it is a bivector field, not a set of scalars which are components of a 2-form, as is the case in Maxwell theory. Also, recall that according to Eq.(201) each one of the four $\mathcal{J}_\beta \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{Cl}(TM)$.*

From Eq.(198) it is not obvious that we must have $\mathcal{F}_{\mathbf{ab}} = 0$ in vacuum, however that is exactly what happens if we take into account Eq.(196) which defines that object. Moreover, $\mathcal{F}_{\mathbf{ab}} = 0$ does not imply that the curvature bivectors $\mathbf{R}_{\mathbf{ab}}$ are null in vacuum. Indeed, in that case, Eq.(196) implies only the identity (valid *only* in vacuum)

$$(\mathbf{e}^c \lrcorner \mathbf{R}_{\mathbf{ac}}) \mathbf{e}_b = (\mathbf{e}^c \lrcorner \mathbf{R}_{\mathbf{bc}}) \mathbf{e}_a. \quad (202)$$

Moreover, recalling definition (Eq.(165)) we have

$$\mathbf{R}_{\mathbf{ab}} = R_{\mathbf{abcd}} \mathbf{e}^c \mathbf{e}^d, \quad (203)$$

and we see that the $\mathbf{R}_{\mathbf{ab}}$ are zero only if the Riemann tensor is null which is not the case in any non trivial general relativistic model.

The important fact that we want to emphasize here is that although eventually interesting, Eq.(198) does not seem (according to our opinion) to contain anything new in it. More precisely, all information given by that equation is already contained in the original Einstein's equation, for indeed it has been obtained from it by simple algebraic manipulations. We state again: According to our view terms like

$$\begin{aligned} \mathcal{F}_{\mathbf{ab}} &= \frac{1}{2}(\mathbf{R}_{\mathbf{ac}} \mathbf{e}^c \mathbf{e}_b + \mathbf{e}_b \mathbf{e}^c \mathbf{R}_{\mathbf{ac}} - \mathbf{e}^c \mathbf{R}_{\mathbf{ac}} \mathbf{e}_b - \mathbf{e}_b \mathbf{R}_{\mathbf{ac}} \mathbf{e}^c) - \frac{1}{2}R(\mathbf{e}_a \mathbf{e}_b - \mathbf{e}_b \mathbf{e}_a), \\ \mathfrak{R}_{\mathbf{ab}} &= (T_a \mathbf{e}_b - \mathbf{e}_b T_a) - \frac{1}{2}R(\mathbf{e}_a \mathbf{e}_b - \mathbf{e}_b \mathbf{e}_a), \\ \mathbf{F}_{\mathbf{ab}} &= \frac{1}{2}R(\mathbf{e}_a \mathbf{e}_b - \mathbf{e}_b \mathbf{e}_a), \end{aligned} \quad (204)$$

are pure gravitational objects. We can see any relationship of any one of these objects with the ones appearing in Maxwell theory. Of course, these objects may eventually be used to formulate interesting equations, like Eq.(198) which are equivalent to Einstein's field equations, but this fact does not seem to us to point to any new Physics.³⁹ Even more, from the mathematical point of view, to find solutions to the new Eq.(198) is certainly as hard as to find solutions to the original Einstein equations.

³⁹Note that $\mathbf{F}_{\mathbf{ab}}$ differs from a factor, namely R from the $\mathbf{F}'_{\mathbf{ab}}$ given by Eq.(70).

7.1 $Sl(2, \mathbb{C})$ Gauge Theory and Sachs Antisymmetric Equation

We discuss in this subsection yet another algebraic exercise. First recall that in section 2 we define the paravector fields,

$$\mathbf{q}_a = \mathbf{e}_a \mathbf{e}_0 = \boldsymbol{\sigma}_a, \quad \check{\mathbf{q}}_a = (-\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_i), \quad \boldsymbol{\sigma}_0 = 1.$$

Recall that⁴⁰

$$\begin{aligned} [D_{e_\rho}, D_{e_\lambda}]e_\mu &= R_{\mu\rho\lambda}^\alpha e_\alpha = -R_{\alpha\mu\rho\lambda}e^\alpha = R_{\mu\alpha\rho\lambda}e^\alpha, \\ R_{\mu\rho\lambda}^\alpha &= \mathcal{R}(e_\mu, \theta^\alpha, e_\rho, e_\lambda). \end{aligned} \quad (205)$$

Then a simple calculation shows that

$$[D_{e_\rho}, D_{e_\lambda}]e_\mu = e_\mu \lrcorner \mathbf{R}_{\rho\lambda} = -\mathbf{R}_{\rho\lambda} \lrcorner e_\mu, \quad (206)$$

$$R_{\mu\alpha\rho\lambda}e^\alpha = \frac{1}{2}(e_\mu \mathbf{R}_{\rho\lambda} - \mathbf{R}_{\rho\lambda} e_\mu). \quad (207)$$

Multiplying Eq.(207) on the left by \mathbf{e}_0 we get, recalling that $\omega_{\mathbf{e}_a}^\dagger = -\mathbf{e}^0 \omega_{\mathbf{e}_a} \mathbf{e}^0$ (Eq.(79)) we get

$$R_{\mu\alpha\rho\lambda} \mathbf{q}^\alpha = \frac{1}{2}(\mathbf{q}_\mu \mathbf{R}_{\rho\lambda}^\dagger + \mathbf{R}_{\rho\lambda} \mathbf{q}_\mu). \quad (208)$$

Now, to derive Sachs⁴¹ Eq.(6.50a) all we need to do is to multiply Eq.(195) on the right by \mathbf{e}^0 and perform some algebraic manipulations. We then get (with *our* normalization) for the equivalent of Einstein's equations using the paravector fields and a coordinate chart $\langle x^\mu \rangle$ covering an open set $U \subset M$, the following equation

$$\mathbf{R}_{\rho\lambda} \mathbf{q}^\lambda + \mathbf{q}^\lambda \mathbf{R}_{\rho\lambda}^\dagger + R \mathbf{q}_\rho = 2 \mathbf{T}_\rho. \quad (209)$$

For the Hermitian conjugate we have

$$-\mathbf{R}_{\rho\lambda}^\dagger \check{\mathbf{q}}^\lambda - \check{\mathbf{q}}^\lambda \mathbf{R}_{\rho\lambda} + R \check{\mathbf{q}}_\rho = 2 \check{\mathbf{T}}_\rho. \quad (210)$$

where as above $\mathbf{R}_{\rho\lambda}$ are the the curvature bivectors given by Eq.(172) and

$$\mathbf{T}_\rho = T_\rho^\mu \mathbf{q}_\mu \in \sec \bigwedge^2 TM \hookrightarrow \mathcal{C}\ell(TM). \quad (211)$$

After that, we multiply Eq.(209) on the right by $\check{\mathbf{q}}_\gamma$ and Eq.(210) on the left by \mathbf{q}_γ ending with two new equations. If we sum them, we get a 'symmetric' equation⁴² completely equivalent to Einstein's equation (from where we started).

⁴⁰In Sachs book he wrote: $[D_{e_\rho}, D_{e_\lambda}]e_\mu = R_{\mu\rho\lambda}^\alpha e_\alpha = +R_{\alpha\mu\rho\lambda}e^\alpha$. This produces some changes in signals in relation to our formulas below. Our Eq.(205) agrees with the conventions in [26].

⁴¹Numeration is from Sachs' book [90].

⁴²Eq.(6.52) in Sachs' book [90].

If we make the difference of the equations we get an antisymmetric equation. The antisymmetric equation can be written, introducing

$$\begin{aligned}\mathbb{F}_{\rho\gamma} = & \frac{1}{2}(\mathbf{R}_{\rho\lambda}\mathbf{q}^\lambda\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\check{\mathbf{q}}^\lambda\mathbf{R}_{\rho\lambda} + \mathbf{q}^\lambda\mathbf{R}_{\rho\lambda}^\dagger\check{\mathbf{q}}_\gamma + \mathbf{q}_\gamma\mathbf{R}_{\rho\lambda}^\dagger\check{\mathbf{q}}^\lambda) \\ & + \frac{1}{2}R(\mathbf{q}_\rho\check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma\check{\mathbf{q}}_\rho)\end{aligned}\quad (212)$$

and

$$\mathbb{J}_\gamma = D_{e_\rho}(\mathbf{T}^\rho\check{\mathbf{q}}_\gamma - \mathbf{q}_\gamma\check{\mathbf{T}}^\rho), \quad (213)$$

as

$$D_{e_\rho}\mathbb{F}_\gamma^\rho = \mathbb{J}_\gamma. \quad (214)$$

Remark 19 *It is important to recall that each one of the six $\mathbb{F}_{\rho\gamma}$ and each one of the four \mathbb{J}_γ are not a set of scalars, but sections of $\mathcal{C}\ell^{(0)}(TM)$. Also, take notice that Eq.(214), of course, is completely equivalent to our Eq.(198). Its matrix translation in $\mathcal{C}\ell^{(0)}(M) \simeq S(M) \otimes_{\mathbb{C}} \bar{S}(M)$ gives Sachs equation (6.52-) in [90] if we take into account his different Sachs different ‘normalization’ of the connection coefficients and the ad hoc factor with dimension of electric charge that he introduced. We cannot see at present any new information encoded in that equations which could be translated in interesting geometrical properties of the manifold, but of course, eventually someone may find that they encode such a useful information.⁴³*

Using the equations, $D_{\mathbf{e}_a}\mathbf{e}_0 = 0$ and $D_{e_\rho}^{\mathbf{S}}\mathbf{q}_\mu = 0$ (respectively, Eq.(88) and Eq.(108) in [?]) and (168) we may verify that

$$D_{e_\rho}^{\mathbf{S}}\mathbb{F}_{\mu\nu} + D_{e_\mu}^{\mathbf{S}}\mathbb{F}_{\nu\rho} + D_{e_\nu}^{\mathbf{S}}\mathbb{F}_{\rho\mu} = 0, \quad (215)$$

where $D_{e_\rho}^{\mathbf{S}}$ is Sachs ‘covariant’ derivative that we discussed in [?]. In [91] Sachs concludes that the last equation implies that there are no magnetic monopoles in nature. Of course, his conclusion would follow from Eq.(215) only if it happened that \mathbb{F}_γ^ρ were the components in a coordinate basis of a 2-form field $F \in \sec \bigwedge^2 T^*M$. However, this is not the case, because as already noted above, this is not the mathematical nature of the \mathbb{F}_γ^ρ . Contrary to what we stated with relation to Eq.(214) we cannot even say that Eq.(215) is really interesting, because it uses a covariant derivative operator, which, as discussed in [?] is not well justified, and in anyway $D_{e_\rho}^{\mathbf{S}} \neq D_{e_\rho}$. We cannot see any relationship of Eq.(215) with the legendary magnetic monopoles.

⁴³Anyway, it seems to us that until the written of the present paper the true mathematical nature of Sachs equations have not been understood, by people that read Sachs books and articles. To endorse our statement, we quote that in Carmeli’s review([23]) of Sachs book, he did not realize that Sachs theory was indeed (as we showed above) a description in the Pauli bundle of a $SL(2, \mathbb{C})$ gauge formulation of Einstein’s theory as described in his own book [22]. Had he disclosed that fact (as we did) he probably had not written that Sachs’ approach was a possible unified field theory of gravitation and electromagnetism.

We thus conclude this section stating that Sachs claims in [90, 91, 92] of having produced an unified field theory of electricity and electromagnetism are not endorsed by our analysis.

8 Energy-Momentum “Conservation” in General Relativity

8.1 Einstein’s Equations in terms of Superpotentials $\star S^a$

In this section we discuss some issues and statements concerning the problem of the energy-momentum conservation in Einstein’s theory, presented with several different formalisms in the literature, which according to our view are very confusing, or even wrong.⁴⁴ To start, recall that from Eq.(184) it follows that

$$d(\star \mathcal{J} - \frac{1}{2}[\omega, \star \mathcal{R}]) = 0, \quad (216)$$

and we could think that this equation could be used to identify a *conservation law* for the energy momentum of matter plus the gravitational field, with $\frac{1}{2}[\omega, \star \mathcal{R}]$ describing a mathematical object related to the energy-momentum of the gravitational field. However, this is not the case, because this term (due to the presence of ω) is gauge dependent. The appearance of a gauge dependent term is a recurrent fact in all known proposed⁴⁵ formulations of a ‘conservation law for energy-momentum’ for Einstein theory. We discuss now some statements found in the literature based on some of that proposed ‘solutions’ to the problem of energy-momentum conservation in General Relativity and say why we think they are unsatisfactory. We also mention a way with which the problem could be satisfactorily solved, but which implies in a departure from the orthodox interpretation of Einstein’s theory.

Now, to keep the mathematics as simple and transparent as possible instead of working with Eq.(216), we work with a more simple (but equivalent) formulation [81, 100] of Einstein’s equation where the gravitational field is described by a set of 2-forms $\star S^a$, $a = 0, 1, 2, 3$ called superpotentials. This approach will permit to identify very quickly certain objects that at first sight seems *appropriate energy-momentum* currents for the gravitational field in Einstein’s theory. The calculations that follows are done in the Clifford algebra of multiforms fields $\mathcal{C}\ell(T^*M)$, something that, as the reader will testify, simplify considerably similar calculations done with traditional methods.

We start again with Einstein’s equations given by Eq.(190), but this time we multiply on the left by $\theta^b \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M)$ getting an equation relating the *Ricci 1-forms* $\mathcal{R}^a = R^a_b \theta^b \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{C}\ell(T^*M)$ with the

⁴⁴Mastering that there is indeed a serious problem, will enable readers to appreciate some of our comments concerning MEG.

⁴⁵At least, the ones known by the authors.

energy-momentum 1-forms $\mathcal{T}^{\mathbf{a}} = T^{\mathbf{a}}_{\mathbf{b}} \theta^{\mathbf{b}} \in \sec \bigwedge^1 T^*M \hookrightarrow \mathcal{Cl}(T^*M)$, i.e.,

$$\mathcal{G}^{\mathbf{a}} = \mathcal{R}^{\mathbf{a}} - \frac{1}{2} R \theta^{\mathbf{a}} = \mathcal{T}^{\mathbf{a}}. \quad (217)$$

We take the dual of this equation,

$$\star \mathcal{G}^{\mathbf{a}} = \star \mathcal{T}^{\mathbf{a}}. \quad (218)$$

Now, we observe that [81, 100] we can write

$$\star \mathcal{G}^{\mathbf{a}} = -d \star \mathcal{S}^{\mathbf{a}} - \star \mathbf{t}^{\mathbf{a}}, \quad (219)$$

where

$$\begin{aligned} \mathcal{S}^{\mathbf{c}} &= -\frac{1}{2} \omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{c}}), \\ \star \mathbf{t}^{\mathbf{c}} &= \frac{1}{2} \omega_{\mathbf{ab}} \wedge [\omega_{\mathbf{d}}^{\mathbf{c}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \omega_{\mathbf{d}}^{\mathbf{b}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{d}} \wedge \theta^{\mathbf{c}})]. \end{aligned} \quad (220)$$

The proof of Eq.(220) follows at once from the fact that

$$\star \mathcal{G}^{\mathbf{d}} = \frac{1}{2} \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}). \quad (221)$$

Indeed, recalling the identities in Eq.(254) we can write

$$\begin{aligned} \frac{1}{2} \mathcal{R}_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) &= -\frac{1}{2} \star [\mathcal{R}_{\mathbf{ab}} \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= -\frac{1}{2} R_{\mathbf{abcd}} \star [(\theta^{\mathbf{c}} \wedge \theta^{\mathbf{d}}) \lrcorner (\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] \\ &= -\star (\mathcal{R}^{\mathbf{d}} - \frac{1}{2} R \theta^{\mathbf{d}}). \end{aligned} \quad (222)$$

On the other hand we have,

$$\begin{aligned} 2 \star \mathcal{G}^{\mathbf{d}} &= d \omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) + \omega_{\mathbf{ac}} \wedge \omega_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ &= d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \omega_{\mathbf{ab}} \wedge d \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ &\quad + \omega_{\mathbf{ac}} \wedge \omega_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ &= d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \omega_{\mathbf{ab}} \wedge \omega_{\mathbf{p}}^{\mathbf{a}} \star(\theta^{\mathbf{p}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ &\quad - \omega_{\mathbf{ab}} \wedge \omega_{\mathbf{p}}^{\mathbf{b}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}}) - \omega_{\mathbf{ab}} \wedge \omega_{\mathbf{p}}^{\mathbf{d}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}})] \\ &\quad + \omega_{\mathbf{ac}} \wedge \omega_{\mathbf{b}}^{\mathbf{c}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}}) \\ &= d[\omega_{\mathbf{ab}} \wedge \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{d}})] - \omega_{\mathbf{ab}} \wedge [\omega_{\mathbf{p}}^{\mathbf{d}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{b}} \wedge \theta^{\mathbf{p}}) + \omega_{\mathbf{p}}^{\mathbf{b}} \star(\theta^{\mathbf{a}} \wedge \theta^{\mathbf{p}} \wedge \theta^{\mathbf{d}})] \\ &= -2(d \star \mathcal{S}^{\mathbf{d}} + \star \mathbf{t}^{\mathbf{d}}). \end{aligned} \quad (223)$$

Now, we can then write Einstein's equation in a very interesting, but *dangerous* form, i.e.:

$$-d \star \mathcal{S}^{\mathbf{a}} = \star \mathcal{T}^{\mathbf{a}} + \star \mathbf{t}^{\mathbf{a}}. \quad (224)$$

In writing Einstein's equations in that way, we have associated to the gravitational field a set of 2-form fields $\star\mathcal{S}^a$ called *superpotentials* that have as sources the currents $(\star\mathcal{T}^a + \star t^a)$. However, superpotentials are not uniquely defined since, e.g., superpotentials $(\star\mathcal{S}^a + \star\alpha^a)$, with $\star\alpha^a$ closed, i.e., $d\star\alpha^a = 0$ give the same second member for Eq.(224).

8.2 Is There Any Energy-Momentum Conservation Law in GR?

Why did we say that Eq.(224) is a dangerous one?

The reason is that (as in the case of Eq.(216)) we can be led to think that we have discovered a conservation law for the energy momentum of matter plus gravitational field, since from Eq.(224) it follows that

$$d(\star\mathcal{T}^a + \star t^a) = 0. \quad (225)$$

This thought however is only an example of wishful thinking, because the $\star t^a$ depends on the connection (see Eq.(220)) and thus are gauge dependent. They do not have the same tensor transformation law as the $\star\mathcal{T}^a$. So, Stokes theorem cannot be used to derive from Eq.(225) conserved quantities that are independent of the gauge, which is clear. However, and this is less known, Stokes theorem, also cannot be used to derive conclusions that are independent of the local coordinate chart used to perform calculations [21]. In fact, the currents $\star t^a$ are nothing more than the old pseudo energy momentum tensor of Einstein in a new dress. Non recognition of this fact can lead to many misunderstandings. We present some of them in what follows, in order to call our readers' attention of potential errors of inference that can be done when we use sophisticated mathematical formalisms without a perfect domain of their contents.

(i) First, it is easy to see that from Eq.(218) it follows that [66]

$$\mathbf{D}^c \star \mathfrak{G} = \mathbf{D}^c \star \mathfrak{T} = 0, \quad (226)$$

where $\star \mathfrak{G} = \mathbf{e}_a \otimes \star \mathcal{G}^a \in \sec TM \otimes \sec \bigwedge^3 T^*M$ and $\star \mathfrak{T} = \mathbf{e}_a \otimes \star \mathcal{T}^a \in \sec TM \otimes \sec \bigwedge^3 T^*M$. Now, in [66] it is written (without proof) a 'Stokes theorem'

$$\boxed{\int_{\text{4-cube}} \mathbf{D}^c \star \mathfrak{T} = \int_{\substack{\text{3 boundary} \\ \text{of this 4-cube}}} \star \mathfrak{T}} \quad (227)$$

We searched in the literature for a proof of Eq.(227) which appears also in many other texts and scientific papers, as e.g., in [29, 104] and could find none, which we can consider as valid. The reason is simply. If expressed in details, e.g., the first member of Eq.(227) reads

$$\int_{\text{4-cube}} \mathbf{e}_a \otimes (d \star \mathcal{T}^a + \omega_b^a \wedge \mathcal{T}^b). \quad (228)$$

and it is necessary to explain what is the meaning (if any) of the integral. Since the integrand is a sum of tensor fields, this integral says that we are *summing* tensors belonging to the tensor spaces of different spacetime points. As, well known, this cannot be done in general, unless there is a way for identification of the tensor spaces at different spacetime points. This requires, of course, the introduction of additional structure on the spacetime representing a given gravitational field, and such extra structure is lacking in Einstein theory. We unfortunately, must conclude that Eq.(227) do not express any conservation law, for it lacks as yet, a precise mathematical meaning.⁴⁶

In Einstein theory possible superpotentials are, of course, the $\star S^a$ that we found above (Eq.(220)), with

$$\star S_c = [-\frac{1}{2}\omega_{ab}\lrcorner(\theta^a \wedge \theta^b \wedge \theta_c)]\theta^5. \quad (229)$$

Then, if we integrate Eq.(224) over a ‘certain finite 3-dimensional volume’, say a ball B , and use Stokes theorem we have

$$P^a = \int_B \star (T^a + t^a) = - \int_{\partial B} \star S^a. \quad (230)$$

In particular the energy or (*inertial mass*) of the gravitational field plus matter generating the field is defined by

$$P^0 = E = m_i = - \lim_{R \rightarrow \infty} \int_{\partial B} \star S^0 \quad (231)$$

(ii) Now, a frequent misunderstanding is the following. Suppose that in a *given* gravitational theory there exists an energy-momentum conservation law for matter plus the gravitational field expressed in the form of Eq.(225), where T^a are the energy-momentum 1-forms of matter and t^a are *true*⁴⁷ energy-momentum 1-forms of the gravitational field. This means that the 3-forms $(\star T^a + \star t^a)$ are closed, i.e., they satisfy Eq.(225). Is this enough to warrant that the energy of a closed universe is zero? Well, that would be the case if starting from Eq.(225) we could jump to an equation like Eq.(224) and then to Eq.(231) (as done, e.g., in [101]). But that sequence of inferences in general cannot be done, for indeed, as it is well known, it is not the case that closed three forms are always exact. Take a closed universe with topology, say $\mathbb{R} \times S^3$. In this case $B = S^3$ and we have $\partial B = \partial S^3 = \emptyset$. Now, as it is well known (see, e.g., [71]), the third de Rham cohomology group of $\mathbb{R} \times S^3$ is $H^3(\mathbb{R} \times S^3) = H^3(S^3) = \mathbb{R}$. Since this group is non trivial it follows that in such manifold closed forms are not exact. Then from Eq.(225) it did not follow the validity of an equation analogous to Eq.(224). So, in that case an equation like Eq.(230) cannot even be written.

⁴⁶Of course, if some could give a mathematical meaning to Eq.(227), we will be glad to be informed of that fact.

⁴⁷This means that the t^a are not pseudo 1-forms, as in Einstein’s theory.

Despite that commentary, keep in mind that in Einstein's theory the energy of a closed universe⁴⁸ if it is given by Eq.(231) is indeed zero, since in that theory the 3-forms $(\star\mathcal{T}^a + \star\mathfrak{t}^a)$ are indeed exact (see Eq.(224)). This means that accepting \mathfrak{t}^a as the energy-momentum 1-form fields of the gravitational field, it follows that gravitational energy *must* be negative in a closed universe.

(iii) But, is the above formalism a consistent one? Given a coordinate chart $\langle x^\mu \rangle$ of the maximal atlas of M , with some algebra we can show that for a gravitational model represented by a diagonal asymptotic flat metric⁴⁹, the inertial mass $E = m_i$ is given by

$$m_i = \lim_{R \rightarrow \infty} \frac{-1}{16\pi} \int_{\partial B} \frac{\partial}{\partial x^\beta} (g_{11}g_{22}g_{33}g^{\alpha\beta}) d\sigma_\alpha, \quad (232)$$

where $\partial B = S^2(R)$ is a 2-sphere of radius R , $(-n_\alpha)$ is the outward unit normal and $d\sigma_\alpha = -R^2 n_\alpha dA$. If we apply Eq.(232) to calculate, e.g., the energy of the Schwarzschild space time⁵⁰ generate by a gravitational mass m , we expect to have one unique and unambiguous result, namely $m_i = m$.

However, as showed in details, e.g., in [21] the calculation of E depends on the spatial coordinate system naturally adapted to the reference frame $Z = \frac{1}{\sqrt{(1-\frac{2m}{r})}} \frac{\partial}{\partial t}$, even if these coordinates produce asymptotically flat metrics. Then, even if in one given chart we may obtain $m_i = m$ there are others where $m_i \neq m$!

Moreover, note also that, as showed above, for a closed universe, Einstein's theory implies on general grounds (once we accept that the \mathfrak{t}^a describes the energy-momentum distribution of the gravitational field) that $m_i = 0$. This result, it is important to quote, does not contradict the so called "positive mass theorems" of, e.g., references [95, 96, 109], because that theorems refers to the total energy of an isolated system. A system of that kind is supposed to be modelled by a Lorentzian spacetime having a spacelike, asymptotically Euclidean hypersurface.⁵¹ However, we want to emphasize here, that although the energy results positive, its value is not unique, since depends on the asymptotically flat coordinates chosen to perform the calculations, as it is clear from the example of the Schwarzschild field, as we already commented above and detailed in [21].

In view of what has been presented above, it is our view that all discourses (based on Einstein's equivalence principle) concerning the use of pseudo-energy momentum tensors as *reasonable* descriptions of energy and momentum of gravitational fields in Einstein's theory are not convincing.

⁴⁸Note that if we suppose that the universe contains spinor fields, then it must be a spin manifold, i.e., it is parallelizable according to Geroch's theorem [51].

⁴⁹A metric is said to be asymptotically flat in given coordinates, if $g_{\mu\nu} = n_{\mu\nu}(1 + O(r^{-k}))$, with $k = 2$ or $k = 1$ depending on the author. See, eg., [95, 96, 106].

⁵⁰For a Scharzschild spacetime we have $g = (1 - \frac{2m}{r}) dt \otimes dt - (1 - \frac{2m}{r})^{-1} dr \otimes dr - r^2(d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)$.

⁵¹The proof also uses as hypothesis the so called energy dominance condition [53]

The fact is: there are *in general* no conservation laws of energy-momentum in General Relativity in general. And, at this point it is better to quote page 98 of Sachs&Wu⁵² [93]:

” As mentioned in section 3.8, conservation laws have a great predictive power. It is a shame to lose the special relativistic total energy conservation law (Section 3.10.2) in general relativity. Many of the attempts to resurrect it are quite interesting; many are simply garbage.”

We quote also Anderson [16]:

” In an interaction that involves the gravitational field a system can loose energy without this energy being transmitted to the gravitational field.”

In General Relativity, we already said, every gravitational field is modelled (module diffeomorphisms) by a Lorentzian spacetime. In the particular case, when this spacetime structure admits a *timelike* Killing vector, we can formulate a law of energy conservation. If the spacetime admits three linearly independent *spacelike* Killing vectors, we have a law of conservation of momentum. The crucial fact to have in mind here is that a general Lorentzian spacetime, does not admits such Killing vectors in general. As one example, we quote that the popular Friedmann-Robertson-Walker expanding universes models do not admit timelike Killing vectors, in general.

At present, the authors know only one possibility of resurrecting a *trust-worthy* conservation law of energy-momentum valid in all circumstances in a theory of the gravitational field that *resembles* General Relativity (in the sense of keeping Einstein’s equation). It consists in reinterpreting that theory as a field theory in flat Minkowski spacetime. Theories of this kind have been proposed in the past by, e.g., Feynman [48], Schwinger [94], Thirring [99] and Weinberg [107, 108] and have been extensively studied by Logunov and collaborators [61, 62]. Another presentation of a theory of that kind, is one where the gravitational field is represented by a distortion field in Minkowski spacetime. A first attempt to such a theory using Clifford bundles has been given in [81]. Another presentation has been given in [58], but that work, which contains many interesting ideas, unfortunately contains also some equivocated statements that make (in our opinion) the theory, as originally presented by that authors invalid. This has been discussed with details in [43].

Before closing this section we observe that recently people think to have find a valid way of having a genuine energy-momentum conservation law in general relativity, by using the so-called *teleparallel* version of that theory [31]. If that is really the case will be analyzed in a sequel paper [88], where we discuss conservation laws in a general Riemann-Cartan spacetime, using Clifford bundle methods.

One of our intentions in writing this section was to leave the reader aware of the *shameful* fact of non energy-momentum conservation in General Relativity when we comment in the next section some papers by Evans&AIAS where they try to explain the functioning of MEG, a ‘motionless electric generator’ that according to those authors pumps energy from the vacuum.

⁵²Note, please, that in this reference Sachs refers to R. K. Sachs and not to M. Sachs.

8.3 “Explanation” of MEG according to AIAS

Our comments on AIAS papers dealing with MEG are the following:

(i) AIAS claim⁵³ that the \mathbf{B}_3 electromagnetic field of their new “ $O(3)$ electrodynamics” is to be identified with \mathbf{F}_{12} (giving by Eq.(204)).

Well, this is a nonsequitur because we already showed above that \mathbf{F}_{12} (Eq.(204)) has nothing to do with electromagnetic fields, it is only a combination of the curvature bivectors, which is a pure gravitational object.

(ii) With that identification AIAS claims that it is the energy of the “electromagnetic” field \mathbf{F}_{12} that makes MEG to work. In that way MEG must be understood as motionless electromagnetic generator that (according to AIAS) pumps energy from the ‘vacuum’ defined by the \mathbf{B}_3 field.

Well, Eq.(204) shows that $\mathbf{F}_{12} = 0$ on the vacuum. It follows that if MEG really works, then it is pumping energy from another source, or it is violating the law of energy-momentum conservation. So, it is unbelievable how Physics journals have published AIAS papers on MEG using arguments as the one just discussed, that are completely wrong.

(iii) We would like to leave it clear here that it is our my opinion that MEG does not work, even if the USPTO granted a patent for that invention, what we considered a very sad and dangerous fact. We already elaborated on this point in the introduction and more discussion on the subject of MEG can be found⁵⁴ at http://groups.yahoo.com/group/free_energy/.

(iv) And what to say about the new electrodynamics of the AIAS group and its \mathbf{B}_3 field?

Well, in [25, 85] we analyzed in deep all known presentations of the “new $O(3)$ electrodynamics” of the AIAS group. It has been proved beyond any doubt that almost all AIAS papers are simply a pot pourri of non sequitur Mathematics and Physics. That is not only our opinion, and the reader is invited (if he become interested on that issue) to read a review of [25] in [17].

Recently ([38]-[41]) Evans is claiming to have produced an unified theory and succeeded in publishing his odd ideas in ISI indexed Physical journals. In the next section we discuss his ‘unified’ theory, showing that it is again, as it is the case of the old Evans&AIAS papers, simply a compendium of nonsense Mathematics and Physics.

(v) And if we are wrong concerning our opinion that MEG does not work?

Well, in that (improbable) case that MEG works, someone can claim that its functioning vindicates the General Theory of Relativity, since as proved in the last section in that theory there is no trustworthy law of energy-momentum conservation. That would be really amazing...

⁵³See the list of their papers related to the subject in the bibliography.

⁵⁴The reader must be aware that there are many nonsequitur posts in this yahoo group, but there are also many serious papers written by serious and competent people.

9 Einstein Field Equations for the Tetrad Fields θ^a

In the main text we gave a Clifford bundle formulation of the field equations of general relativity in a form that resembles a $Sl(2, \mathbb{C})$ gauge theory and also a formulation in terms of a set of 2-form fields $\star \mathcal{S}^a$. Here we want to *recall* yet another face of Einstein's equations, i.e., we show how to write the field equations directly for the tetrad fields θ^a in such a way that the obtained equations are equivalent to Einstein's field equations. This is done in order to compare the correct equations for that objects which some other equations proposed for these objects that appeared recently in the literature (and which will be discussed below). Before proceeding, we mention that, of course, we could write analogous (and equivalent) equations for the dual tetrads e_a .

As shown in details in papers [81, 97] the correct wave like equations satisfied by the θ^a are⁵⁵:

$$-(\partial \cdot \partial)\theta^a + \partial \wedge (\partial \cdot \theta^a) + \partial \lrcorner (\partial \wedge \theta^a) = \mathcal{T}^a - \frac{1}{2}T\theta^a. \quad (233)$$

In Eq.(233), $\mathcal{T}^a = T_b^a \theta^b \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ are the energy momentum 1-form fields and $T = T_a^a = -R = -R_a^a$, with T_{ab} the energy momentum tensor of matter. When θ^a is an exact differential, and in this case we write $\theta^a \mapsto \theta^\mu = dx^\mu$ and if the coordinate functions are harmonic, i.e., $\delta\theta^\mu = -\partial\theta^\mu = 0$, Eq.(233) becomes

$$\square\theta^\mu + \frac{1}{2}R\theta^\mu = -\mathcal{T}^\mu, \quad (234)$$

where we have used Eq. (A.16),

$$(\partial \cdot \partial) = \square \quad (235)$$

i.e., $\partial \cdot \partial$ is the (covariant) D'Alembertian operator.

In Eq.(233) $\partial = \theta^a D_{e_a} = \partial \wedge + \partial \lrcorner = d - \delta$ is the Dirac (like) operator acting on sections of the Clifford bundle $\mathcal{C}\ell(T^*M)$ defined in the previous Appendix.

With these formulas we can write

$$\begin{aligned} \partial^2 &= \partial \cdot \partial + \partial \wedge \partial, \\ \partial \wedge \partial &= -\partial \cdot \partial + \partial \wedge \partial \lrcorner + \partial \lrcorner \partial \wedge, \end{aligned} \quad (236)$$

with

$$\begin{aligned} \partial \cdot \partial &= \eta^{ab}(D_{e_a} D_{e_b} - \omega_{ab}^c D_{e_c}), \\ \partial \wedge \partial &= \theta^a \wedge \theta^b (D_{e_a} D_{e_b} - \omega_{ab}^c D_{e_c}). \end{aligned} \quad (237)$$

⁵⁵Of course, there are analogous equations for the e_a [54], where in that case, the Dirac operator must be defined (in an obvious way) as acting on sections of the Clifford bundle of multivectors, that has been introduced in section 3.

Note that $D_{e_a}\theta^b = -\omega_{ac}^b\theta^c$ and a somewhat long, but simple calculation ⁵⁶ shows that

$$(\partial \wedge \partial)\theta^a = \mathcal{R}^a, \quad (238)$$

where, as already defined, $\mathcal{R}^a = R_b^a\theta^b$ are the Ricci 1-forms. We also observe (that for the best of our knowledge) $\partial \wedge \partial$ that has been named the Ricci operator in [97] has no analogue in classical differential geometry.

Note that Eq. (233) can be written after some algebra as

$$\mathcal{R}^\mu - \frac{1}{2}R\theta^\mu = T^\mu, \quad (239)$$

with $\mathcal{R}^\mu = R_\nu^\mu dx^\nu$ and $T^\mu = T_\nu^\mu dx^\nu$, $\theta^\mu = dx^\mu$ in a coordinate chart of the maximal atlas of M covering an open set $U \subset M$.

We are now prepared to make some crucial comments concerning some recent papers ([27],[39]-[42]).

(i) In ([27],[39]-[42]) authors claims that the \mathbf{e}_a , $a = 0, 1, 2, 3$ satisfy the equations

$$(\square + T)\mathbf{e}_a = 0.$$

They thought to have produced a valid derivation for that equations. We will not comment on that derivation here. Enough is to say that if that equation was true it would imply that $(\square + T)\theta^a = 0$. This is not the case. Indeed, as a careful reader may verify, the true equation satisfied by any one of the θ^a is Eq.(233).

(ii) We quote that authors of [39, 40, 41] explicitly wrote several times that the "electromagnetic potential"⁵⁷ \mathbf{A} in their theory (a 1-form with values in a vector space) satisfies the following wave equation,

$$(\square + T)\mathbf{A} = 0.$$

Now, this equation cannot be correct even for the usual $U(1)$ gauge potential of classical electrodynamics ⁵⁸ $A \in \sec \bigwedge^1 T^*M \subset \sec \mathcal{C}\ell(T^*M)$. Indeed, in vacuum Maxwell equation reads (see Eq.(261))

$$\partial F = 0, \quad (240)$$

where $F = \partial A = \partial \wedge A = dA$, if we work in the Lorenz gauge $\partial \cdot A = \partial \lrcorner A = -\delta A = 0$. Now, since we have according to Eq.(??) that $\partial^2 = -(d\delta + \delta d)$, we get

$$\partial^2 A = 0. \quad (241)$$

⁵⁶The calculation is done in detail in [81, 97].

⁵⁷In [39, 40, 41] authors do identify their "electromagnetic potential" with the bivector valued connection 1-form ω that we introduced in section above. As we explained with details this cannot be done because that quantity is related to gravitation, not electromagnetism.

⁵⁸Which must be one of the gauge components of the gauge field.

A simple calculation then shows that in the coordinate basis introduced above we have,

$$(\partial^2 A)_\alpha = g^{\mu\nu} D_\mu D_\nu A_\alpha + R_\alpha^\nu A_\nu \quad (242)$$

and we see that Eq.(241) reads in components

$$D_\alpha D^\alpha A_\mu + R_\mu^\nu A_\nu = 0. \quad (243)$$

Eq.(243) can be found, e.g., in Eddington's book [32] on page 175. Take also notice that in Einstein theory in vacuum the term $R_\mu^\nu A_\nu = 0$.

Finally we make a single comment on reference [27], because this paper is related to Sachs 'unified' theory in the sense that authors try to identify Sachs 'electromagnetic' field (discussed in the main text) with a supposedly existing longitudinal electromagnetic field predict by their theory. Well, on [27] we can read at the beginning of section 1.1:

"The antisymmetrized form of special relativity [1] has spacetime metric given by the enlarged structure

$$\eta^{\mu\nu} = \frac{1}{2} (\sigma^\mu \sigma^{\nu*} + \sigma^\nu \sigma^{\mu*}), \quad (1.1.)$$

where σ^μ are the Pauli matrices satisfying a clifford (sic) algebra

$$\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu},$$

which are represented by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.2)$$

The $*$ operator denotes quaternion conjugation, which translates to a spatial parity transformation."

Well, we comment as follows: the $*$ is not really defined anywhere in [27]. If it refers to a spatial parity operation, we infer that $\sigma^{0*} = \sigma^0$ and $\sigma^{i*} = -\sigma^i$. Also, $\eta^{\mu\nu}$ is not defined, but Eq.(3.5) of [27] make us to infer that $\eta^{\mu\nu} = \text{diag}(1, -1, -1, 1)$. In that case Eq.(1.1) above is true (if the first member is understood as $\eta^{\mu\nu} \sigma^0$) but the equation $\{\sigma^\mu, \sigma^\nu\} = 2\delta^{\mu\nu}$ is false. Enough is to see that $\{\sigma^0, \sigma^i\} = 2\sigma^i \neq 2\delta^{0i}$.

10 Conclusions

In this paper we introduced the concept of Clifford valued differential forms, which are sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. We showed how this theory can be used to produce a very elegant description the theory of linear connections, where a given linear connection is represented by a bivector valued 1-form. Crucial to the program was the introduction the notion of the exterior covariant derivative of sections of $\mathcal{C}\ell(TM) \otimes \bigwedge T^*M$. Our *natural* definitions parallel in a noticeable way the formalism of the theory of connections in a principal bundle and

the covariant derivative operators acting on associate bundles to that principal bundle. We identified Cartan curvature 2-forms and *curvature bivectors*. The curvature 2-forms satisfy Cartan's second structure equation and the curvature bivectors satisfy equations in analogy with equations of gauge theories. This immediately suggest to write Einstein's theory in that formalism, something that has already been done and extensively studied in the past. However, we did not enter into the details of that theory in this paper. We only discussed the relation between the nonhomogeneous $Sl(2, \mathbb{C})$ gauge equation satisfied by the curvature bivector and the problem of the energy-momentum 'conservation' in General Relativity, and also between that theory and M. Sachs 'unified' field theory as described in [90, 91].

To make a complete analysis of M. Sachs 'unified' field theory we also recalled the concept of covariant derivatives of spinor fields, when these objects are represented as sections of real spinor bundles ([59, 67, 83]) and study how this theory has as matrix representative the standard spinor fields (dotted and undotted) already introduced long ago, see, e.g., [22, 75, 76, 77]. What was new in our approach is that we identify a possible profound physical meaning concerning some of the rules used in the standard formulation of the (matrix) formulation of spinor fields, e.g., why the covariant derivative of the Pauli matrices must be null. Those rules implies in constraints for the geometry of the spacetime manifold. A possible realization of that constraints is one where the fields defining a global tetrad must be such that \mathbf{e}_0 is a geodesic field and the \mathbf{e}_i are Fermi transported (i.e., are not rotating relative to the "fixed stars") along each integral line of \mathbf{e}_0 . For the best of our knowledge this important fact is here disclosed for the first time.

We use our formalism to discuss several issues in presentations of gravitational theory and other theories. In particular, we scrutinized Sachs "unified" the theory as discussed recently in [91, 92] and as originally introduced in [90]. It is really difficult to believe that after that more than 40 years Sachs succeeded in publishing his doubtable results without anyone denouncing his errors. The case is worth to have in mind when we realized that Sachs has more than 900 citations in the Science Citation Index. Some one may say: who cares? Well, we cared, for reasons mainly described in the introduction, and here we showed that there are some crucial mathematical errors in that theory. To start, [90, 91, 92] identified erroneously his basic variables q_μ as being (matrix representations) of quaternion fields. Well, they are not. The real mathematical structure of these objects is that they are matrix representations of particular sections of the even Clifford bundle of multivectors $\mathcal{Cl}(TM)$ as we proved in section 2. Next we show that the identification of a 'new' antisymmetric field $\mathbb{F}_{\alpha\beta}$ (Eq.(212)) in his theory is indeed nothing more than the identification of some combinations of the curvature bivectors⁵⁹, an object that appears naturally when we try to formulate Einstein's gravitational theory as a $Sl(2, \mathbb{C})$ gauge theory. In that way, any tentative of identifying $\mathbb{F}_{\alpha\beta}$ with any kind of electromagnetic field as

⁵⁹The curvature bivectors are physically and mathematically equivalent to the Cartan curvature 2-forms, since they carry the same information. This statement is obvious from our study in section 4.

did by Sachs in [90, 91] is clearly wrong. We also present the wave like equations solved by the (co)tetrad fields⁶⁰ θ^a . Equipped with the correct mathematical formulation of some sophisticated notions of modern Physics theories we identified *fatal mathematical flaws* in several papers by Evans&AIAS⁶¹ that use Sachs ‘unified’ theory. In a series of papers, quoted in the bibliography Evans&AIAS claims that MEG works with the energy of the \mathbf{B}_3 field that they identified with the field \mathbf{F}_{12} (given by Eq.(204)) that appears in Sachs theory. They thought, following Sachs, that that field represents an electromagnetic field. It is amazing how referees of that papers could accept that argument, for in vacuum $\mathbf{F}_{12} = 0$ (see Eq.(204)). Also, as already said \mathbf{F}_{12} and also \mathbb{F}_{12} re not electromagnetic fields. However, since there are no conservation laws of energy-momentum in general relativity, if MEG works⁶², maybe it is only demonstrating this aspect of General Relativity, that may authors on the subject try (hard) to hide under the carpet.

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A Clifford Bundles $\mathcal{Cl}(T^*M)$ and $\mathcal{Cl}(TM)$

Let $\mathcal{L} = (M, g, D, \tau_g, \uparrow)$ be a Lorentzian spacetime. This means that (M, g, τ_g, \uparrow) is a four dimensional Lorentzian manifold, time oriented by \uparrow and spacetime oriented by τ_g , with $M \simeq \mathbb{R}^4$ and $g \in \text{sec}(T^*M \times T^*M)$ being a Lorentzian metric of signature (1,3). T^*M [TM] is the cotangent [tangent] bundle. $T^*M = \cup_{x \in M} T_x^*M$, $TM = \cup_{x \in M} T_xM$, and $T_xM \simeq T_x^*M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space [93].. D is the Levi-Civita connection of g , i.e., $Dg = 0$, $\mathcal{R}(D) = 0$. Also $\Theta(D) = 0$, \mathcal{R} and Θ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{Cl}(T^*M)$ is the bundle of algebras⁶³ $\mathcal{Cl}(T^*M) = \cup_{x \in M} \mathcal{Cl}(T_x^*M)$, where $\forall x \in M, \mathcal{Cl}(T_x^*M) = \mathbb{R}_{1,3}$, the so-called *spacetime algebra* [60]. Locally as a linear space over the real field R , $\mathcal{Cl}(T_x^*M)$ is isomorphic to the Cartan algebra $\bigwedge(T_x^*M)$ of the cotangent space and $\bigwedge T_x^*M = \sum_{k=0}^4 \bigwedge^k T_x^*M$, where $\bigwedge^k T_x^*M$ is the $\binom{4}{k}$ -dimensional space of k -forms. The Cartan bundle $\bigwedge T^*M = \cup_{x \in M} \bigwedge T_x^*M$ can then be thought [59] as “imbedded” in $\mathcal{Cl}(T^*M)$. In this way sections of $\mathcal{Cl}(T^*M)$ can be represented as a sum of nonhomogeneous differential forms. Let $\{\mathbf{e}_a\} \in \text{sec} TM$, ($\mathbf{a} = 0, 1, 2, 3$) be an orthonormal basis $g(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab} = \text{diag}(1, -1, -1, -1)$ and let $\{\theta^a\} \in \text{sec} \bigwedge^1 T^*M \hookrightarrow \text{sec} \mathcal{Cl}(T^*M)$ be the dual basis. Moreover, we denote by g^{-1} the metric in the cotangent bundle.

⁶⁰The set $\{\theta^a\}$ is the dual basis of $\{\mathbf{e}_a\}$.

⁶¹Recall that Evans is as quoted as Sachs, according to the Science Citation Index...

⁶²We stated above our opinion that despite MEG is a patented device it does not work.

⁶³We can show using the definitions of section 5 that $\mathcal{Cl}(T^*M)$ is a vector bundle associated with the *orthonormal frame bundle*, i.e., $\mathcal{C}(M) = P_{SO_+(1,3)} \times_{ad} Cl_{1,3}$. Details about this construction can be found, e.g., in [67].

An analogous construction can be done for the tangent space. The corresponding Clifford bundle is denoted $\mathcal{Cl}(TM)$ and their sections are called multivector fields. All formulas presented below for $\mathcal{Cl}(T^*M)$ have a corresponding in $\mathcal{Cl}(TM)$ and this fact has been used in the text.

A.1 Clifford product, scalar contraction and exterior products

The fundamental *Clifford product* (in what follows to be denoted by juxtaposition of symbols) is generated by $\theta^a \theta^b + \theta^b \theta^a = 2\eta^{ab}$ and if $\mathcal{C} \in \sec \mathcal{Cl}(T^*M)$ we have

$$C = s + v_a \theta^a + \frac{1}{2!} b_{cd} \theta^c \theta^d + \frac{1}{3!} a_{abc} \theta^a \theta^b \theta^c + p \theta^5, \quad (244)$$

where $\theta^5 = \theta^0 \theta^1 \theta^2 \theta^3$ is the volume element and $s, v_a, b_{cd}, a_{abc}, p \in \sec \bigwedge^0 T^*M \subset \sec \mathcal{Cl}(T^*M)$.

Let $A_r, \in \sec \bigwedge^r T^*M \hookrightarrow \sec \mathcal{Cl}(T^*M), B_s \in \sec \bigwedge^s T^*M \hookrightarrow \sec \mathcal{Cl}(T^*M)$. For $r = s = 1$, we define the *scalar product* as follows:

For $a, b \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{Cl}(T^*M)$,

$$a \cdot b = \frac{1}{2}(ab + ba) = g^{-1}(a, b). \quad (245)$$

We also define the *exterior product* ($\forall r, s = 0, 1, 2, 3$) by

$$\begin{aligned} A_r \wedge B_s &= \langle A_r B_s \rangle_{r+s}, \\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r \end{aligned} \quad (246)$$

where $\langle \rangle_k$ is the component in the subspace $\bigwedge^k T^*M$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{Cl}(T^*M)$.

For $A_r = a_1 \wedge \dots \wedge a_r, B_r = b_1 \wedge \dots \wedge b_r$, the *scalar product* is defined as

$$\begin{aligned} A_r \cdot B_r &= (a_1 \wedge \dots \wedge a_r) \cdot (b_1 \wedge \dots \wedge b_r) \\ &= \det \begin{bmatrix} a_1 \cdot b_1 & \dots & a_1 \cdot b_k \\ \dots & \dots & \dots \\ a_k \cdot b_1 & \dots & a_k \cdot b_k \end{bmatrix}. \end{aligned} \quad (247)$$

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$ then $A_r \cdot B_s = 0$.

For $r \leq s, A_r = a_1 \wedge \dots \wedge a_r, B_s = b_1 \wedge \dots \wedge b_s$ we define the *left contraction* by

$$\lrcorner : (A_r, B_s) \mapsto A_r \lrcorner B_s = \sum_{i_1 < \dots < i_r} \epsilon_{1 \dots r}^{i_1 \dots i_r} (a_1 \wedge \dots \wedge a_r) \cdot (b_{i_1} \wedge \dots \wedge b_{i_r}) \sim b_{i_r+1} \wedge \dots \wedge b_{i_s}, \quad (248)$$

where \sim denotes the reverse mapping (*reversion*)

$$\sim: \sec \bigwedge^p T^*M \ni a_1 \wedge \dots \wedge a_p \mapsto a_p \wedge \dots \wedge a_1, \quad (249)$$

and extended by linearity to all sections of $\mathcal{C}\ell(T^*M)$. We agree that for $\alpha, \beta \in \sec \bigwedge^0 T^*M$ the contraction is the ordinary (pointwise) product in the real field and that if $\alpha \in \sec \bigwedge^0 T^*M$, $A_r \in \sec \bigwedge^r T^*M$, $B_s \in \sec \bigwedge^s T^*M$ then $(\alpha A_r) \lrcorner B_s = A_r \lrcorner (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}\ell(T^*M)$, i.e., for $A, B \in \sec \mathcal{C}\ell(T^*M)$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, r \leq s. \quad (250)$$

It is also necessary to introduce in $\mathcal{C}\ell(T^*M)$ the operator of *right contraction* denoted by \lrcorner . The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r \in \sec \bigwedge^r T^*M$, $B_s \in \sec \bigwedge^s T^*M$ then $A_r \lrcorner (\alpha B_s) = (\alpha A_r) \lrcorner B_s$.

A.2 Some useful formulas

The main formulas used in the Clifford calculus in the main text can be obtained from the following ones, where $a \in \sec \bigwedge^1 T^*M$ and $A_r \in \sec \bigwedge^r T^*M$, $B_s \in \sec \bigwedge^s T^*M$:

$$\begin{aligned} a B_s &= a \lrcorner B_s + a \wedge B_s, B_s a = B_s \lrcorner a + B_s \wedge a, \\ a \lrcorner B_s &= \frac{1}{2}(a B_s - (-)^s B_s a), \\ A_r \lrcorner B_s &= (-)^{r(s-1)} B_s \lrcorner A_r, \\ a \wedge B_s &= \frac{1}{2}(a B_s + (-)^s B_s a), \\ A_r B_s &= \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + \dots + \langle A_r B_s \rangle_{|r+s|} \\ &= \sum_{k=0}^m \langle A_r B_s \rangle_{|r-s|+2k}, \quad m = \frac{1}{2}(r+s-|r-s|). \end{aligned} \quad (251)$$

A.3 Hodge star operator

Let \star be the usual Hodge star operator $\star: \bigwedge^k T^*M \rightarrow \bigwedge^{4-k} T^*M$. If $B \in \sec \bigwedge^k T^*M$, $A \in \sec \bigwedge^{4-k} T^*M$ and $\tau \in \sec \bigwedge^4 T^*M$ is the volume form, then $\star B$ is defined by

$$A \wedge \star B = (A \cdot B) \tau.$$

Then we can show that if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$ we have

$$\star A_p = \widetilde{A_p} \theta^5. \quad (252)$$

This equation is enough to prove very easily the following identities (which are used in the main text):

$$\begin{aligned}
A_r \wedge \star B_s &= B_s \wedge \star A_r; \quad r = s, \\
A_r \lrcorner \star B_s &= B_s \lrcorner \star A_r; \quad r + s = 4, \\
A_r \wedge \star B_s &= (-1)^{r(s-1)} \star (\tilde{A}_r \lrcorner B_s); \quad r \leq s, \\
A_r \lrcorner \star B_s &= (-1)^{rs} \star (\tilde{A}_r \wedge B_s); \quad r + s \leq 4
\end{aligned} \tag{254}$$

Let d and δ be respectively the differential and Hodge codifferential operators acting on sections of $\bigwedge^p T^*M$. If $\omega_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(T^*M)$, then $\delta\omega_p = (-1)^p \star^{-1} d \star \omega_p$, with $\star^{-1}\star = \text{identity}$. When applied to a p -form we have

$$\star^{-1} = (-1)^{p(4-p)+1} \star.$$

A.4 Action of $D_{\mathbf{e}_a}$ on Sections of $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$

Let $D_{\mathbf{e}_a}$ be the Levi-Civita covariant derivative operator acting on sections of the tensor bundle. It can be easily shown (see, e.g., [28]) that $D_{\mathbf{e}_a}$ is also a covariant derivative operator on the Clifford bundles $\mathcal{C}\ell(TM)$ and $\mathcal{C}\ell(T^*M)$.

Now, if $A_p \in \sec \bigwedge^p T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ we can show, very easily by explicitly performing the calculations⁶⁴ that

$$D_{\mathbf{e}_a} A_p = \partial_{\mathbf{e}_a} A_p + \frac{1}{2} [\omega_{\mathbf{e}_a}, A_p], \tag{255}$$

where the $\omega_{\mathbf{e}_a} \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}\ell(M)$ may be called *Clifford connection 2-forms*. They are given by:

$$\omega_{\mathbf{e}_a} = \frac{1}{2} \omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \theta_{\mathbf{c}} = \frac{1}{2} \omega_{\mathbf{a}}^{\mathbf{bc}} \theta_{\mathbf{b}} \wedge \theta_{\mathbf{c}}, \tag{256}$$

where (in standard notation)

$$D_{\mathbf{e}_a} \theta_{\mathbf{b}} = \omega_{\mathbf{ab}}^{\mathbf{c}} \theta_{\mathbf{c}}, \quad D_{\mathbf{e}_a} \theta^{\mathbf{b}} = -\omega_{\mathbf{ac}}^{\mathbf{b}} \theta^{\mathbf{c}}, \quad \omega_{\mathbf{a}}^{\mathbf{bc}} = -\omega_{\mathbf{a}}^{\mathbf{cb}} \tag{257}$$

An analogous formula to Eq.(255) is valid for the covariant derivative of sections of $\mathcal{C}\ell(TM)$ and they are used in several places in the main text.

A.5 Dirac Operator, Differential and Codifferential

The Dirac operator acting on sections of $\mathcal{C}\ell(T^*M)$ is the invariant first order differential operator

$$\partial = \theta^{\mathbf{a}} D_{\mathbf{e}_a}, \tag{258}$$

and we can show(see, e.g., [81]) the very important result:

$$\partial = \partial \wedge + \partial \lrcorner = d - \delta. \tag{259}$$

⁶⁴A derivation of this formula from the genral theory of connections can be found in [67].

The square of the Dirac operator ∂^2 is called the *Hodge Laplacian*. It is not to be confused with the covariant D'Alembertian which is given by $\square = \partial \cdot \partial$. The following identities are used in the text

$$\begin{aligned} dd &= \delta\delta = 0, \\ d\partial^2 &= \partial^2 d; \quad \delta\partial^2 = \partial^2 \delta, \\ \delta\star &= (-1)^{p+1} \star d; \quad \star\delta = (-1)^p \star d, \\ d\delta\star &= \star d\delta; \quad \star d\delta = \delta d\star; \quad \star\partial^2 = \partial^2 \star \end{aligned} \quad (260)$$

A.6 Maxwell Equation

Maxwell equations in the Clifford bundle of differential forms resume in one single equation. Indeed, if $F \in \sec \bigwedge^2 T^*M \subset \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic field and $J_e \in \sec \bigwedge^1 T^*M \subset \sec \mathcal{C}\ell(T^*M)$ is the electromagnetic current, we have Maxwell equation⁶⁵:

$$\partial F = J_e. \quad (261)$$

Eq.(261) is equivalent to the pair of equations

$$dF = 0, \quad (262)$$

$$\delta F = -J_e. \quad (263)$$

Eq.(262) is called the homogenous equation and Eq.(263) is called the non-homogeneous equation. Note that it can be written also as:

$$d\star F = -\star J_e. \quad (264)$$

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